

EQUIVARIANT SUBCONVEX BOUNDS FOR HECKE–MAASS FORMS ON SEMISIMPLE GROUPS

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ABSTRACT. In this paper, we generalize existing spherical subconvex bounds for Hecke–Maass forms on semisimple groups in the eigenvalue aspect to non-spherical situations for cocompact lattices using the spectral function of an elliptic operator.

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1. INTRODUCTION

Let M be a closed connected Riemannian manifold M of dimension d and $P_0 : C^\infty(M) \rightarrow L^2(M)$ an elliptic classical pseudodifferential operator on M of degree m , where $C^\infty(M)$ denotes the space of smooth functions on M and $L^2(M)$ the space of square-integrable functions on M . Assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P with the m -th Sobolev space as domain, and let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j \geq 0}$ repeated according to their multiplicity. By a classical result of Avacumovic, Levitan, and Hörmander [1, 20, 12] one has for any $j \in \mathbb{N}$ the *convex bound*¹

$$(1.1) \quad \|\phi_j\|_\infty \ll \lambda_j^{\frac{d-1}{2m}}.$$

If the $\{\phi_j\}_{j \geq 0}$ are also eigenfunctions of a commuting family of differential operators on M , this bound can be improved. Thus, assume that M carries an isometric action of a compact connected Lie group K such that all orbits have the same dimension κ . Suppose further that P commutes with the action of K , and that the cosphere bundles $S_x^*M := \{(x, \xi) \in T^*M : p(x, \xi) = 1\}$ are strictly convex, where T^*M denotes the cotangent bundle of M and $p(x, \xi)$ the principal symbol of P_0 . For an arbitrary class of irreducible unitary representations $\sigma \in \widehat{K}$ of dimension d_σ , it was shown in [26] that the *equivariant convex bound*

$$(1.2) \quad \|\phi_j\|_\infty \ll d_\sigma \cdot \lambda_j^{\frac{d-\kappa-1}{2m}}, \quad \phi_j \in L_\sigma^2(M),$$

holds for any eigenfunction in the σ -isotypic component $L_\sigma^2(M)$ of the Peter–Weyl decomposition of $L^2(M)$. Similarly, let G be a semisimple real Lie group, K a maximal compact subgroup of G , $\Gamma \subset G$ a lattice, and $Y := \Gamma \backslash G/K$ the corresponding locally symmetric space of dimension d and rank r . If

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¹Here and in what follows we shall write $a \ll_\gamma b$ for two real numbers a and b and a variable γ , if there exists a constant $C_\gamma > 0$ depending only on γ such that $|a| \leq C_\gamma b$.

$\{\psi_j\}_{j \geq 0}$ constitutes an orthonormal basis in $L^2(Y)$ of simultaneous eigenfunctions of the full ring of invariant differential operators on Y , which is isomorphic to a finitely generated polynomial ring in r variables and contains the Beltrami-Laplace operator Δ , Sarnak [27] was able to show the *spherical convex bound*

$$(1.3) \quad \|\psi_j|_\Omega\|_\infty \ll \lambda_j^{\frac{d-r}{4}}$$

for arbitrary compacta $\Omega \subset Y$, $\{\lambda_j\}_{j \geq 0}$ being the eigenvalues of Δ .

The bounds (1.1) and (1.2) are known to be sharp on the standard d -sphere, but for the Beltrami-Laplace operator Δ on a compact congruence hyperbolic surface $\Gamma \backslash \mathbb{H}$, given as a quotient of the complex upper half plane \mathbb{H} by the group of units Γ in an order in an indefinite quaternion division algebra over \mathbb{Q} , Iwaniec and Sarnak [16] were able to strengthen the bound (1.1) and prove for any $\varepsilon > 0$ and $j \in \mathbb{N}$ the substantially stronger *spherical subconvex bound*

$$(1.4) \quad \|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24} + \varepsilon},$$

where $d = m = 2$, provided that the ϕ_j are also eigenfunctions of the ring of Hecke operators on $L^2(\Gamma \backslash \mathbb{H})$. In general, functions on a manifold which are simultaneous eigenfunctions of an invariant differential operator and some ring of Hecke operators are known as *Hecke–Maass forms*. More generally, if H is a semisimple algebraic group over \mathbb{Q} satisfying some conditions, $\Gamma \subset H(\mathbb{Q})$ an arithmetic congruence lattice, and $G = H(\mathbb{R})$, Marshall [21] was able to strengthen (1.3) and prove spherical subconvex bounds for Hecke–Maass forms in $L^2(Y)$ of the form

$$(1.5) \quad \|\psi_j|_\Omega\|_\infty \ll \lambda_j^{\frac{d-r}{4} - \delta}$$

for some $\delta > 0$ and arbitrary compacta $\Omega \subset Y$, generalizing previous work of Blomer-Maga [3, 4] and Blomer-Pohl [5], among others. In fact, for negatively curved manifolds, much better bounds are expected to hold generically, the bound (1.4) being the strongest known bound up to now.

The goal of this paper is to prove subconvex bounds in non-spherical cases and sharpen the bound (1.2) in case that the eigenfunctions ϕ_j are Hecke–Maass forms. As our first main result, we extend the bounds (1.4) to non-trivial K -types. Thus, consider the groups $G := \mathrm{SL}(2, \mathbb{R})$, $K := \mathrm{SO}(2)$, and let \mathcal{R} be an Eichler order in an indefinite division quaternion algebra A over \mathbb{Q} . Denote by $N(x)$ the reduced norm of an element $x \in A$, and write $\mathcal{R}(m) := \{\alpha \in \mathcal{R} : N(\alpha) = m\}$ for any $m \in \mathbb{N}$. Choose an embedding $\theta : \sqcup_{m=1}^\infty \mathcal{R}(m) \rightarrow G$, and set $\Gamma := \theta(\mathcal{R}(1))$. Then Γ constitutes a congruence arithmetic subgroup, and $\Gamma \backslash \mathbb{H} \simeq \Gamma \backslash G/K$ becomes a compact hyperbolic surface. Now, let χ be a Nebentypus character on Γ , and denote by $L_\chi^2(\Gamma \backslash G)$ the Hilbert space of measurable functions on G such that

$$f(\gamma x) = \chi(\gamma) f(x), \quad \gamma \in \Gamma, x \in G, \quad \|f\| := \left(\int_{\Gamma \backslash G} |f(x)|^2 dx \right)^{1/2} < \infty.$$

The space $L_\chi^2(\Gamma \backslash G)$ can be regarded as a closed subspace in $L^2(\Gamma_\chi \backslash G)$, where $\Gamma_\chi := \ker \chi$. Identifying $\mathcal{R}(n)$ with its image $\theta(\mathcal{R}(n))$ for each n prime to a fixed natural number which depends only on \mathcal{R} , the finite cosets $\mathcal{R}(1) \backslash \mathcal{R}(n)$ give rise to the Hecke operators

$$(T_n^\chi f)(x) := \sum_{\alpha \in \Gamma \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

where $f \in L_\chi^2(\Gamma \backslash G)$, and we wrote $\alpha \equiv \Gamma \alpha$ for short. Now, fix an arbitrary K -type $\sigma \in \widehat{K}$, and denote by $L_{\sigma, \chi}^2(\Gamma \backslash G)$ the σ -isotypic component of $L_\chi^2(\Gamma \backslash G)$. Then, we show in Theorem 5.7 that for any orthonormal basis $\{\phi_j\}_{j \geq 0}$ of $L^2(\Gamma_\chi \backslash G)$ consisting of Hecke–Maass forms with Beltrami–Laplace eigenvalues $\{\lambda_j\}_{j \geq 0}$ one has the upper bound

$$(1.6) \quad \|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{11}{48} + \varepsilon}, \quad \phi_j \in L_{\sigma, \chi}^2(\Gamma \backslash G),$$

for arbitrary small $\varepsilon > 0$.

Our second main result concerns bounds of the form (1.5). As before, let H be a semisimple algebraic group of adjoint type over \mathbb{Q} . Write $\mathbb{A}_{\mathrm{fin}}$ for the finite adele ring of \mathbb{Q} and $\mathbb{A} := \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ for

the adèle ring. Choosing an open compact subgroup K_0 in $H(\mathbb{A}_{\text{fin}})$, we obtain an arithmetic subgroup $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$ in the semisimple Lie group $G := H(\mathbb{R})$. Assume that the intersection of Γ with each connected component of G is not empty and that $H(\mathbb{Q}) \backslash H(\mathbb{A})$ is compact, so that $\Gamma \backslash G$ is also compact. From the point of view of automorphic representations, there is a suitable family of Hecke operators on $L^2(\Gamma \backslash G)$ for each finite place p of \mathbb{Q} that is prime to a fixed natural number depending on H and K_0 , given by an unramified Hecke algebra over \mathbb{Q}_p . Now, let K be a maximal connected compact subgroup of G , $\sigma \in \widehat{K}$, and $\{\phi_j\}_{j \geq 0}$ an orthonormal basis of $L^2(\Gamma \backslash G)$ consisting of Hecke–Maass forms with respect to an invariant differential operator of order m with spectral values $\{\lambda_j\}_{j \geq 0}$ and strictly convex cosphere bundles. Then, if H satisfies the condition “ K -small” or “complex” of [21], we show in Theorem 6.3 that

$$(1.7) \quad \|\phi_j\|_\infty \ll \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}, \quad \phi_j \in L^2_\sigma(\Gamma \backslash G),$$

for some constant $\delta > 0$. An example would be given by $H = \text{PGL}(1, D)$, where D is any central division algebra D of index n over \mathbb{Q} , and $G = \text{PGL}(n, \mathbb{R})$.

Let us briefly say a few words about the methods employed, and P be an elliptic pseudodifferential operator on a closed Riemannian manifold M as above. Our main tool for deriving our results is the *spectral function* $e(x, y, \mu)$ of the m -th root $Q := \sqrt[m]{P}$ of P given by

$$e(x, y, \mu) := \sum_{\mu_j \leq \mu} \phi_j(x) \overline{\phi_j(y)} \in C^\infty(M \times M), \quad \mu \in \mathbb{R}, \quad \mu_j := \sqrt[m]{\lambda_j}.$$

In fact, in the spherical situations [16, 3, 4, 5, 21] examined before, an important role is played by the spectral expansion of point pair invariants on \mathbb{H} [16, Eq. (1.3)] or asymptotics of spherical functions [21, Eq. (17)]. Since these are not available in our setting,² we shall instead base our analysis on the spectral expansion of $e(x, y, \mu)$ itself and asymptotics of

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu),$$

which represents the Schwartz kernel of the spectral projection s_μ onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$. If M carries an effective and isometric action of a compact connected Lie group K and $\sigma \in \widehat{K}$, denote by Π_σ the projector onto the σ -isotypic component in the Peter-Weyl decomposition of $L^2(M)$. In order to show the equivariant convex bounds (1.2), an asymptotic formula for the Schwartz kernel of $s_\mu \circ \Pi_\sigma$, or rather of $\tilde{s}_\mu \circ \Pi_\sigma$, where \tilde{s}_μ represents certain smooth approximation to s_μ , was derived in [26, Corollary 2.2. and Theorem 3.3] in a neighbourhood of the diagonal relying on the theory of Fourier integral operators. Now, let G be a semisimple Lie group with finite center, Γ a discrete cocompact subgroup, and K a maximal compact subgroup of G . For $\chi \in \widehat{\Gamma}$, introduce on $L^2(\Gamma_\chi \backslash G)$ the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$

$$(\mathcal{T}_{\Gamma\beta\Gamma}^\chi f)(x) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

where β belongs to a certain set containing the commensurator $C(\Gamma)$ of Γ . Based on the asymptotics for the kernel of $\tilde{s}_\mu \circ \Pi_\sigma$ mentioned above, we deduce in Corollary 4.3 for any $\delta > 0$ and some constant $C > 0$ the bound

$$\begin{aligned} K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &\ll \mu^{\dim G/K - 1} M(x, \beta, \delta) + \mu^{(\dim G/K - 1)/2} \int_\delta^C s^{-1/2} dM(s) \\ &\quad + \mu^{(\dim G/K - 1)/2 - 1} \int_\delta^C s^{-1/2 - 1/(d-1)} dM(s) \end{aligned}$$

for the Schwartz kernel of $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma$, where we introduced the lattice point counting function

$$M(\delta) := M(x, \beta, \delta) := \# \{ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma : \text{dist}(xK, \alpha xK)^{\dim G - 1} < \delta \}$$

²Compare also [22, Section 4.4].

given in terms of the distance function on the Riemannian symmetric space G/K . From this, we obtain the equivariant subconvex bounds (1.6) and (1.7) by using known uniform upper bounds [16, 21] for $M(x, \beta, \delta)$ in combination with arithmetic amplification.

Let us close this introduction by noticing that there exist several variants of the bounds (1.4). Thus, in [16, Appendix], the non compact hyperbolic surface $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ is considered. On the other hand, bounds in the level aspect are shown in [29] for compact locally symmetric spaces of arithmetic type, while hybrid bounds in the eigenvalue and level aspect are derived in [2, 30] and others for the modular surfaces $\Gamma_0(N) \backslash \mathbb{H}$. It is likely that our work can be extended to these settings, and we intend to deal with these questions in a future paper.

This paper is structured as follows. In Section 2, we introduce Hecke operators with characters on semisimple Lie groups with finite center, in Section 3 we give a description of the reduced spectral function of an invariant elliptic operator by means of Fourier integral operators, and explain how equivariant convex bounds can be deduced from it. Based on these results, we derive spectral asymptotics for kernels of Hecke operators in Section 4. Relying on the latter, we finally prove equivariant subconvex bounds for arithmetic congruence lattices in $\mathrm{SL}(2, \mathbb{R})$ and a large class of semisimple algebraic groups in Sections 5 and 6, respectively.

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2. HECKE OPERATORS ON SEMISIMPLE LIE GROUPS WITH CHARACTER

To introduce our setting, let G be a connected semisimple Lie group with finite center and Lie algebra \mathfrak{g} , and Γ a discrete cocompact subgroup. Consider a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of \mathfrak{g} , and denote the maximal compact subgroup of G with Lie algebra \mathfrak{k} by K . Choose a left-invariant metric on G given by an $\mathrm{Ad}(K)$ -invariant bilinear form on \mathfrak{g} . It induces a Riemannian metric on G/K , by which G/K becomes a Riemannian symmetric space of non-positive sectional curvature. Let $\chi \in \widehat{\Gamma}$ be an irreducible character of Γ , corresponding to an equivalence class of 1-dimensional unitary representations of Γ , such that $\Gamma_\chi := \ker \chi$ is a subgroup of finite index in Γ . The quotient $\Gamma_\chi \backslash G$ is a compact manifold without boundary, and by requiring that the projection $G \rightarrow \Gamma_\chi \backslash G$ is a Riemannian submersion, we obtain a Riemannian structure on $\Gamma_\chi \backslash G$. Similarly, the Riemannian structure on G/K induces a Riemannian metric on $\Gamma_\chi \backslash G/K$, becoming a locally symmetric space of negative curvature. In what follows, we will simultaneously consider the cases

$$X := G \quad \text{or} \quad G/K, \quad M := \Gamma_\chi \backslash X.$$

In order to introduce Hecke operators on $\Gamma \backslash X$ we follow [11, Section 2], and consider the commensurator

$$C(\Gamma) := \{g \in G \mid \Gamma \text{ is commensurable with } g^{-1}\Gamma g\}$$

of Γ , where we say that two subgroups Γ_1 and Γ_2 are *commensurable* iff the indices $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are finite. Let $\beta \in C(\Gamma)$. Since the mapping

$$(\Gamma \cap \beta^{-1}\Gamma\beta) \backslash \Gamma \ni (\Gamma \cap \beta^{-1}\Gamma\beta)\gamma \mapsto \Gamma\beta\gamma \in \Gamma \backslash \Gamma\beta\Gamma$$

is bijective, the double coset $\Gamma\beta\Gamma$ is a finite union of right cosets of Γ , that is, there exist representative elements $\beta_1, \beta_2, \dots, \beta_t$ in $\Gamma\beta\Gamma$ such that

$$\Gamma\beta\Gamma = \bigsqcup_{j=1}^t \Gamma\beta_j.$$

One can then associate to each double coset a linear operator $T_{\Gamma\beta\Gamma}$ on $L^2(\Gamma\backslash X)$ by setting

$$T_{\Gamma\beta\Gamma} : L^2(\Gamma\backslash X) \longrightarrow L^2(\Gamma\backslash X), \quad (T_{\Gamma\beta\Gamma}f)(x) := \sum_{j=1}^t f(\beta_j \cdot x).$$

$T_{\Gamma\beta\Gamma}$ is called a *Hecke operator*, and since the above sum clearly does not depend on the choice of the representatives β_j , we may also write

$$(T_{\Gamma\beta\Gamma}f)(x) = \sum_{\alpha \in \Gamma\backslash\Gamma\beta\Gamma} f(\alpha \cdot x).$$

If a subset U of $C(\Gamma)$ is decomposed into a finite disjoint union of double cosets of Γ , a linear operator T_U can be defined in the same manner according to

$$(2.1) \quad T_U := \sum_{k=1}^u T_{\Gamma\beta_k\Gamma} \quad U = \bigsqcup_{k=1}^u \Gamma\beta_k\Gamma, \quad \beta_k \in C(\Gamma).$$

More generally, one can introduce Hecke operators as follows. Write $H(\Gamma, C(\Gamma))$ for the space of left and right Γ -invariant \mathbb{C} -valued functions h on $C(\Gamma)$ such that the support of h is included in a finite union of double Γ -cosets. Endowed with the convolution product

$$h_1 * h_2(x) := \sum_{y \in \Gamma\backslash C(\Gamma)} h_1(y) h_2(xy^{-1}), \quad h_1, h_2 \in H(\Gamma, C(\Gamma)),$$

$H(\Gamma, C(\Gamma))$ becomes an associative algebra over \mathbb{C} with the characteristic function 1_Γ of Γ as unit element. For each $h \in H(\Gamma, C(\Gamma))$, a linear operator T_h on $L^2(\Gamma\backslash X)$ can then be defined by

$$(T_h f)(x) := \sum_{\alpha \in \Gamma\backslash C(\Gamma)} h(\alpha) f(\alpha \cdot x),$$

and one has $T_{h_1 * h_2} = T_{h_1} \circ T_{h_2}$. If U is as above and h is the characteristic function of U , then it is obvious that T_h equals T_U . We call $H(\Gamma, C(\Gamma))$ the *Hecke algebra* and refer the reader to [11, Section 2] for details.

Next, we introduce Hecke operators with characters. Assume that G is a subgroup of another group G' . Let Ξ be a sub-semigroup of G' containing Γ . We suppose that χ can be extended to Ξ , and that there exists a homomorphism $\psi : \Xi \rightarrow C(\Gamma)$ such that $\psi|_\Gamma$ is the identity map, and $\alpha x \alpha^{-1} = \psi(\alpha) x \psi(\alpha)^{-1}$ holds for any $\alpha \in \Xi$, $x \in G$. In particular, for $\alpha, \beta \in \Xi$ we have $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$, and the inverse element α^{-1} does not always belong to Ξ .

Example 2.1. One of the main examples we are having in mind is

$$G' := \mathrm{GL}(n, \mathbb{R}), \quad \Xi := \{\alpha = (\alpha_{ij}) \in M(n, \mathbb{Z}) \mid \det(\alpha) > 0, \alpha_{j1} \equiv 0 \pmod{N} \ (2 \leq j \leq n)\},$$

$$\chi((\alpha_{ij})) := \omega(\alpha_{11}), \quad G := \mathrm{SL}(n, \mathbb{R}), \quad \Gamma := G \cap \Xi, \quad \psi(\alpha) := \det(\alpha)^{-1/n} \alpha,$$

where ω is a Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^\times$.

Let us now define a left action of Ξ on G by setting $\alpha \cdot x := \psi(\alpha)x$. For a fixed $\beta \in \Xi$ we can then consider the linear operator

$$(2.2) \quad \mathcal{T}_{\Gamma\beta\Gamma}^\chi : L^2(\Gamma_\chi\backslash X) \longrightarrow L^2(\Gamma_\chi\backslash X), \quad (\mathcal{T}_{\Gamma\beta\Gamma}^\chi f)(x) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi\backslash\Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

where we took into account that $\Gamma_\chi\backslash\Gamma\beta\Gamma \subset \Xi$. By definition we have $\mathcal{T}_{\Gamma\beta\Gamma}^\chi = T_h$ for some $h \in H(\Gamma_\chi, C(\Gamma))$ satisfying $h(\gamma_1 x \gamma_2) = \overline{\chi(\gamma_1 \gamma_2)} h(x)$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in C(\Gamma)$. Furthermore, for given $\beta_j \in \Xi$ and $h_j \in H(\Gamma_\chi, C(\Gamma))$ with $T_{h_j} = \mathcal{T}_{\Gamma\beta_j\Gamma}^\chi$, the convolution $h_1 * h_2$ also satisfies the latter condition. Thus, there exist elements $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in \Xi$ such that

$$(2.3) \quad \mathcal{T}_{\Gamma\beta_1\Gamma}^\chi \circ \mathcal{T}_{\Gamma\beta_2\Gamma}^\chi = \sum_{u=1}^l a_u \mathcal{T}_{\Gamma\alpha_u\Gamma}^\chi.$$

Next, denote by $L^2_\chi(\Gamma \backslash X)$ the Hilbert space of measurable functions on X such that

$$(2.4) \quad f(\gamma x) = \chi(\gamma) f(x), \quad \gamma \in \Gamma, \quad x \in X,$$

and

$$(2.5) \quad \|f\| := \left(\int_{\Gamma \backslash X} |f(x)|^2 dx \right)^{1/2} < \infty,$$

which is well-defined since $|\chi(\gamma)| = 1$ for $\gamma \in \Gamma$, compare [23, p. 228].³ Notice that $f \in L^2_\chi(\Gamma \backslash X)$ implies $|f| \in L^2(\Gamma \backslash X)$. If χ is trivial, then $L^2_\chi(\Gamma \backslash X) = L^2(\Gamma \backslash X)$. Since Γ_χ is a normal subset of Γ we have

$$(2.6) \quad L^2(\Gamma_\chi \backslash X) \cong \bigoplus_{\chi' \in \widehat{\Gamma/\Gamma_\chi}} L^2_{\chi'}(\Gamma \backslash X),$$

where we regard $\widehat{\Gamma/\Gamma_\chi}$ as a subset of $\widehat{\Gamma}$, compare [23, Lemma 4.3.1]. In particular, because $\chi \in \widehat{\Gamma/\Gamma_\chi}$, $L^2_\chi(\Gamma \backslash X)$ is a closed subspace in $L^2(\Gamma_\chi \backslash X)$, and for a fixed $\beta \in \Xi$, the operator $\mathcal{T}^\chi_{\Gamma\beta\Gamma}$ restricts to the linear operator

$$(2.7) \quad T^\chi_{\Gamma\beta\Gamma} : L^2_\chi(\Gamma \backslash X) \longrightarrow L^2_\chi(\Gamma \backslash X), \quad (T^\chi_{\Gamma\beta\Gamma} f)(x) := (\mathcal{T}^\chi_{\Gamma\beta\Gamma} f)(x) = \sum_{\alpha \in \Gamma \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x).$$

Notice that for each χ' in $\widehat{\Gamma/\Gamma_\chi}$ with $\chi' \neq \chi$ and each function $f \in L_{\chi'}(\Gamma \backslash X)$ we have

$$(2.8) \quad \mathcal{T}^\chi_{\Gamma\beta\Gamma} f(x) = [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha_1 \in \Gamma_\chi \backslash \Gamma} \sum_{\alpha_2 \in \Gamma \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha_1 \alpha_2)} \chi'(\alpha_1) f(\alpha_2 \cdot x) = 0$$

by the orthogonality relations for characters. Further, the projection of $L^2(\Gamma_\chi \backslash X)$ onto $L^2_\chi(\Gamma \backslash X)$ is given by the Hecke operator $T_{h(\chi)}$, where $h(\chi) \in H(\Gamma_\chi, \Xi)$ is given by

$$(2.9) \quad h(\chi) : x \longmapsto [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma} \overline{\chi(\alpha)} \mathbf{1}_{\Gamma_\chi \alpha}(x),$$

$\mathbf{1}_{\Gamma_\chi \alpha}$ being the characteristic function of the coset $\Gamma_\chi \alpha$. Thus, one obtains the commutative diagram

$$(2.10) \quad \begin{array}{ccc} L^2(\Gamma_\chi \backslash X) & \xrightarrow{\mathcal{T}^\chi_{\Gamma\beta\Gamma}} & L^2(\Gamma_\chi \backslash X) \\ \downarrow T_{h(\chi)} & & \downarrow T_{h(\chi)} \\ L^2_\chi(\Gamma \backslash X) & \xrightarrow{\mathcal{T}^\chi_{\Gamma\beta\Gamma}} & L^2_\chi(\Gamma \backslash X) \end{array}$$

and in view of (2.8) we have $T_{h(\chi)} \circ T^\chi_{\Gamma\beta\Gamma} \circ T_{h(\chi)} = \mathcal{T}^\chi_{\Gamma\beta\Gamma}$.

3. THE REDUCED SPECTRAL FUNCTION AND EQUIVARIANT CONVEX BOUNDS FOR EIGENFUNCTIONS

In [26], asymptotics for the reduced spectral function of an invariant elliptic operator were deduced within the theory of Fourier integral operators, yielding equivariant convex bounds for eigenfunctions. In what follows, we shall briefly recall the main arguments, and provide the results that will be needed later in our analysis.

³Note that for $\gamma \in \Gamma_\chi$ condition (2.4) reads $f(\gamma x) = f(x)$. Therefore, instead of $L^2_\chi(\Gamma \backslash X)$ one could also consider the closed subspace of $L^2(\Gamma_\chi \backslash X)$ that consists of functions satisfying (2.4).

3.1. The spectral function of an elliptic operator. Let M be a closed connected Riemannian manifold M of dimension d and P_0 an elliptic classical pseudodifferential operator on M of degree m , which is assumed to be positive and symmetric. Denote its unique self-adjoint extension by P , and let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j \geq 0}$ repeated according to their multiplicity. Let $p(x, \xi)$ be the principal symbol of P_0 , which is strictly positive and homogeneous in ξ of degree m as a function on $T^*M \setminus \{0\}$, that is, the cotangent bundle of M without the zero section. Here and in what follows (x, ξ) denotes an element in $T^*Y \simeq Y \times \mathbb{R}^d$ with respect to the canonical trivialization of the cotangent bundle over a chart domain $Y \subset M$. Consider further the m -th root $Q := \sqrt[m]{P}$ of P given by the spectral theorem. It is well known that Q is a classical pseudodifferential operator of order 1 with principal symbol $q(x, \xi) := \sqrt[m]{p(x, \xi)}$ and the first Sobolev space as domain. Again, Q has discrete spectrum, and its eigenvalues are given by $\mu_j := \sqrt[m]{\lambda_j}$. The spectral function $e(x, y, \lambda)$ of P can then be described by studying the spectral function of Q , which in terms of the basis $\{\phi_j\}$ is given by

$$(3.1) \quad e(x, y, \mu) := \sum_{\mu_j \leq \mu} \phi_j(x) \overline{\phi_j(y)},$$

and belongs to $C^\infty(M \times M)$ as a function of x and y for any $\mu \in \mathbb{R}$. Let s_μ be the spectral projection onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$, and denote its Schwartz kernel by

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu).$$

To obtain an asymptotic description of the spectral function of Q , one first derives a description of $s_\mu(x, y)$ by approximating s_μ by Fourier integral operators. To do so, let $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ be such that $\varrho(0) = 1$ and $\text{supp } \hat{\varrho} \in (-\delta/2, \delta/2)$ for an arbitrarily small $\delta > 0$, and define the approximate spectral projection operator

$$(3.2) \quad \tilde{s}_\mu u := \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) E_j u, \quad u \in L^2(M),$$

where E_j denotes the orthogonal projection onto the subspace spanned by ϕ_j . Clearly, $K_{\tilde{s}_\mu}(x, y) := \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)} \in C^\infty(M \times M)$ constitutes the kernel of \tilde{s}_μ . Now, notice that for $\mu, \tau \in \mathbb{R}$ one has

$$\varrho(\mu - \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t) e^{-it\tau} e^{it\mu} dt,$$

where $\hat{\varrho}(t)$ denotes the Fourier transform of ϱ , so that for $u \in L^2(M)$ we obtain

$$(3.3) \quad \tilde{s}_\mu u = \frac{1}{2\pi} \sum_{j=0}^{\infty} \int_{\mathbb{R}} \hat{\varrho}(t) e^{it\mu} e^{-it\mu_j} dt E_j u = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t) e^{it\mu} U(t) u dt,$$

where $U(t)$ stands for the one-parameter group

$$U(t) := e^{-itQ} = \int e^{-it\mu} dE_\mu^Q, \quad t \in \mathbb{R},$$

of unitary operators in $L^2(M)$ given by the Fourier transform of the spectral measure, $\{E_\mu^Q\}$ being a spectral resolution of Q . The central result of Hörmander [12] then says that $U(t) : L^2(M) \rightarrow L^2(M)$ can be approximated by Fourier integral operators.

More precisely, let $\{(\kappa_\iota, Y_\iota)\}_{\iota \in I}$, $\kappa_\iota : Y_\iota \xrightarrow{\sim} \tilde{Y}_\iota \subset \mathbb{R}^d$, be an atlas for M , $\{f_\iota\}$ a corresponding partition of unity and $\hat{v}(\eta) := \mathcal{F}(v)(\eta) := \int_{\mathbb{R}^d} e^{-i\langle \tilde{y}, \eta \rangle} v(\tilde{y}) d\tilde{y}$ the Fourier transform of $v \in C_c^\infty(\tilde{Y}_\iota)$. Write $d\eta := d\eta/(2\pi)^d$, and introduce on \tilde{Y}_ι the operator

$$[\tilde{U}_\iota(t)v](\tilde{x}) := \int_{\mathbb{R}^d} e^{i\psi_\iota(t, \tilde{x}, \eta)} a_\iota(t, \tilde{x}, \eta) \hat{v}(\eta) d\eta,$$

where $a_\iota \in S_{\text{phg}}^0$ is a classical polyhomogeneous symbol satisfying $a_\iota(0, \tilde{x}, \eta) = 1$ and ψ_ι the defining phase function given as the solution of the Hamilton-Jacobi equation

$$\frac{\partial \psi_\iota}{\partial t} + q\left(x, \frac{\partial \psi_\iota}{\partial \tilde{x}}\right) = 0, \quad \psi_\iota(0, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle,$$

see [14, Page 254]. Let us remark that ψ_ι is homogeneous in η of degree 1, so that Taylor expansion for small t gives

$$\psi_\iota(t, \tilde{x}, \eta) = \psi_\iota(0, \tilde{x}, \eta) + t \frac{\partial \psi_\iota}{\partial t}(0, \tilde{x}, \eta) + O(t^2|\eta|) = \langle \tilde{x}, \eta \rangle - tq_\iota(\tilde{x}, \eta) + O(t^2|\eta|),$$

where we wrote $q_\iota(\tilde{x}, \eta) := q(\kappa_\iota^{-1}(\tilde{x}), \eta)$. In other words, there exists a smooth function ζ_ι which is homogeneous in η of degree 1 and satisfies

$$\psi_\iota(t, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle - t\zeta_\iota(t, \tilde{x}, \eta), \quad \zeta_\iota(0, \tilde{x}, \eta) = q_\iota(\tilde{x}, \eta).$$

Let now $\bar{U}_\iota(t)u := [\tilde{U}_\iota(t)(u \circ \kappa_\iota^{-1})] \circ \kappa_\iota$ for any $u \in C_c^\infty(Y_\iota)$. Consider further test functions $\bar{f}_\iota \in C_c^\infty(Y_\iota)$ satisfying $\bar{f}_\iota \equiv 1$ on $\text{supp } f_\iota$, and define

$$\bar{U}(t) := \sum_\iota F_\iota \bar{U}_\iota(t) \bar{F}_\iota,$$

where F_ι, \bar{F}_ι denote the multiplication operators with f_ι and \bar{f}_ι , respectively. Then Hörmander showed that for small $|t|$

$$(3.4) \quad \mathcal{R}(t) := U(t) - \bar{U}(t) \text{ is an operator with smooth kernel,}$$

compare [10, Page 134] and [28, Theorem 20.1]; in particular, the kernel $\mathcal{R}_t(x, y)$ of $\mathcal{R}(t)$ is smooth as a function of t .

Approximating in (3.3) the operator $U(t)$ by $\bar{U}(t)$, asymptotic formulae for the kernels of \tilde{s}_μ and s_μ follow, from which Weyl's law for the spectral function of Q and bounds for eigenfunctions can be deduced, since $\|s_\mu\|_{L^2 \rightarrow L^\infty}^2 \equiv \sup_{x \in M} s_\mu(x, x)$, yielding for any eigenfunction the convex bound

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-1}{2m}}.$$

3.2. Equivariant convex bounds for eigenfunctions. Keeping the notation of Section 3.1, assume now that M carries an effective and isometric action of a compact connected Lie group K , and consider the right regular representation π of K on $L^2(M)$ with corresponding Peter-Weyl decomposition

$$(3.5) \quad L^2(M) = \bigoplus_{\sigma \in \hat{K}} L_\sigma^2(M), \quad L_\sigma^2(M) := \Pi_\sigma L^2(M),$$

where \hat{K} denotes the unitary dual of K and

$$\Pi_\sigma := d_\sigma \int_K \overline{\sigma(k)} \pi(k) dk$$

the orthogonal projector onto the σ -isotypic component, dk being Haar measure and d_σ the dimension of an irreducible representation of K in the class $\sigma \in \hat{K}$. Further, suppose that P commutes with π , and that the orthonormal basis $\{\phi_j\}_{j \geq 0}$ is compatible with the decomposition (3.5) in the sense that each ϕ_j lies in some $L_\sigma^2(M)$. Then every eigenspace of P is invariant under π , and decomposes into irreducible K -modules spanned by eigenfunctions. In order to study eigenfunctions of P of a certain K -type, one is interested in the spectral function of the operator $Q_\sigma := \Pi_\sigma \circ Q \circ \Pi_\sigma = \Pi_\sigma \circ Q = Q \circ \Pi_\sigma$ given by

$$(3.6) \quad e_\sigma(x, y, \mu) = \sum_{\mu_j \leq \mu, \phi_j \in L_\sigma^2(M)} \phi_j(x) \overline{\phi_j(y)}.$$

For this, one considers the composition $s_\mu \circ \Pi_\sigma$, or rather $\tilde{s}_\mu \circ \Pi_\sigma$, whose kernel has the spectral expansion

$$(3.7) \quad K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \sum_{j \geq 0, \phi_j \in L_\sigma^2(M)} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)}.$$

It was shown in [26, Eq. (2.8)] that by approximating $U(t)$ in (3.3) by the Fourier integral operator $\bar{U}(t)$ one obtains a description for the kernel of $\tilde{s}_\mu \circ \Pi_\sigma$ as a double oscillatory integral

$$(3.8) \quad K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \frac{\mu^d d_\sigma}{(2\pi)^{d+1}} \sum_l \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} I_l(\mu, R, t, x, y) dR dt$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y , where

$$\begin{aligned} I_l(\mu, R, t, x, y) &:= \int_K \int_{\Sigma_{l,x}^{R,t}} e^{i\mu\Phi_{l,x,y}(\omega,k)} \hat{\varrho}(t) \overline{\sigma(k)} f_l(x) \\ &\quad \cdot a_l(t, \kappa_l(x), \mu\omega) \bar{f}_l(y \cdot k) b(q(x, \omega)) J_l(k, y) d\Sigma_{l,x}^{R,t}(\omega) dk, \\ \Phi_{l,x,y}(\omega, k) &:= \langle \kappa_l(x) - \kappa_l(y \cdot k), \omega \rangle, \end{aligned}$$

and

$$(3.9) \quad \Sigma_{l,x}^{R,t} := \{ \omega \in \mathbb{R}^d \mid \zeta_l(t, \kappa_l(x), \omega) = R \},$$

while $0 \leq b \in C_c^\infty(1/2, 3/2)$ is a test function such that $b \equiv 1$ in a neighborhood of 1 and $J_l(k, y)$ is a Jacobian. Here $d\Sigma_{l,x}^{R,t}(\omega)$ denotes the quotient of Lebesgue measure in \mathbb{R}^d by Lebesgue measure in \mathbb{R} with respect to $\zeta_l(t, \tilde{x}, \omega)$. Furthermore, for sufficiently small $\delta > 0$ one can assume that the R -integration is over a compact set, and R and t are close to 1 and 0, respectively. We now have the following

Proposition 3.1. *Suppose that K acts on M with orbits of the same dimension $\kappa \leq d-1$ and that the cospheres $S_x^*M := \{(x, \xi) \in T^*M \mid p(x, \xi) = 1\}$ are strictly convex.⁴ Then, for any fixed $x, y \in M$, $\sigma \in \hat{K}$, and $\tilde{N} \in \mathbb{N}$ the expansion*

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \left(\frac{\mu}{2\pi} \right)^{d - \frac{\varepsilon_{x,y}}{2} - 1} \frac{d_\sigma}{2\pi} \left[\sum_{k=0}^{\tilde{N}-1} \mathcal{L}_k(x, y) \mu^{-k} + \mathcal{R}_{\tilde{N}}(x, y, \mu) \right]$$

holds as $\mu \rightarrow +\infty$, where $\mathcal{R}_{\tilde{N}}(x, y, \mu) = O_{x,y}(\mu^{-\tilde{N}})$ and

$$\varepsilon_{x,y} := \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ d-1+\kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients $\mathcal{L}_k(x, y)$ and the remainder term can be computed explicitly; if $y \notin \mathcal{O}_x$, they satisfy the bounds

$$\begin{aligned} \mathcal{L}_k(x, y) &\ll \text{dist}(x, \mathcal{O}_y)^{-\frac{d-1}{2} - k(d-1)}, \\ \mathcal{R}_{\tilde{N}}(x, y, \mu) &\ll \text{dist}(x, \mathcal{O}_y)^{-\frac{d-1}{2} - \tilde{N}(d-1)} \mu^{-\tilde{N}}, \end{aligned}$$

and are uniformly bounded in x and y if $y \in \mathcal{O}_x$.

Remark 3.2.

(1) Proposition 3.1 implies that

$$\begin{aligned} (3.10) \quad |K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)| &\leq \sqrt{\sum_{j \geq 0, \phi_j \in L_\sigma^2(M)} \varrho(\mu - \mu_j) |\phi_j(x)|^2} \sqrt{\sum_{j \geq 0, \phi_j \in L_\sigma^2(M)} \varrho(\mu - \mu_j) |\phi_j(y)|^2} \\ &= \sqrt{K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)} \sqrt{K_{\tilde{s}_\mu \circ \Pi_\sigma}(y, y)} = O(d_\sigma \mu^{d-\kappa-1}) \end{aligned}$$

uniformly in $x, y \in M$ by Cauchy-Schwarz.

⁴This is the case, for example, if $P_0 = \Delta$ equals the Laplace-Beltrami operator, since then $p(x, \xi) = \|\xi\|_x^2$.

- (2) Note that the asymptotics of Proposition 3.1 in the case $y \notin \mathcal{O}_x$ are only meaningful if $\text{dist}(x, \mathcal{O}_y)^{-d-1} \leq \mu$.

Proof of Proposition 3.1. An asymptotic expansion of $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)$ was given in [26, Proposition 4.2]. To obtain an asymptotic expansion off the diagonal from (3.8), we shall first apply the stationary phase theorem to the integrals $I_\iota(\mu, R, t, x, y)$, and then to the (R, t) -integral. Write $\mathcal{O}_x := x \cdot K$ for the K -orbit through $x \in M$. If $x \notin Y_\iota$ or $\mathcal{O}_y \cap Y_\iota = \emptyset$, $I_\iota(\mu, R, t, x, y) = 0$. Otherwise, [26, Theorem 3.3] implies for sufficiently small Y_ι , fixed $R, t \in \mathbb{R}$, and any $\tilde{N} \in \mathbb{N}$ the asymptotic expansion

$$(3.11) \quad I_\iota(\mu, R, t, x, y) = \left(\frac{2\pi}{\mu}\right)^{\frac{\text{codim Crit } \Phi_{\iota, x, y}}{2}} e^{i\mu \Phi_{\iota, x, y}^0(R, t)} \left[\sum_{j=0}^{\tilde{N}-1} \mathcal{L}_\iota^j(R, t, x, y) \mu^{-j} + \tilde{\mathcal{R}}_{\iota, \tilde{N}}(R, t, x, y, \mu) \right],$$

where $\text{Crit } \Phi_{\iota, x, y}$ denotes the critical set of $\Phi_{\iota, x, y}$, and

$$\text{codim Crit } \Phi_{\iota, x, y} = \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ d-1+\kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients $\mathcal{L}_\iota^j(R, t, x, y)$ and the remainder term are given by distributions depending smoothly on R, t with support in $\text{Crit } \Phi_{\iota, x, y}$ and $\Sigma_{\iota, x}^{R, t} \times K$, respectively. Furthermore, they and their derivatives with respect to R, t satisfy for $y \notin \mathcal{O}_x$ the bounds

$$\begin{aligned} |\partial_{R, t}^\alpha \mathcal{L}_\iota^j(R, t, x, y)| &\leq \tilde{C}_j \text{dist}(x, \mathcal{O}_y)^{-\frac{d-1}{2}-j(d-1)}, \\ |\partial_{R, t}^\alpha \tilde{\mathcal{R}}_{\iota, \tilde{N}}(R, t, x, y, \mu)| &\leq \tilde{C}_{\tilde{N}} \text{dist}(x, \mathcal{O}_y)^{-\frac{d-1}{2}-\tilde{N}(d-1)} \mu^{-\tilde{N}} \end{aligned}$$

for suitable constants $\tilde{C}_j, \tilde{C}_{\tilde{N}} > 0$, while for $y \in \mathcal{O}_x$ one has the uniform bounds

$$|\mathcal{L}_\iota^j(R, t, x, y)| \leq \tilde{C}_j, \quad |\tilde{\mathcal{R}}_{\iota, \tilde{N}}(R, t, x, y, \mu)| \leq \tilde{C}_{\tilde{N}} \mu^{-\tilde{N}}.$$

Finally,

$$\Phi_{\iota, x, y}^0(R, t) = R c_{x, y}(t), \quad c_{x, y}(t) = \pm \frac{\|\kappa_\iota(x) - \kappa_\iota(y \cdot k)\|}{\text{grad}_\eta \zeta_\iota(t, \kappa_\iota(x), \omega)},$$

denotes the constant value(s) of $\Phi_{\iota, x, y}$ on (the components) of its critical set, where (ω, k) is some point in $\text{Crit } \Phi_{\iota, x, y}$. If $y \in \mathcal{O}_x$ one has $\Phi_{\iota, x, y}^0(R, t) = 0$. Note that $a_\iota \in S_{phg}^0$ is a classical symbol of order 0, so that

$$|\partial_\omega^\xi a_\iota(t, \kappa_\iota(x), \mu\omega)| = |\mu|^{|\alpha|} |(\partial_\omega^\xi a_\iota)(t, \kappa_\iota(x), \mu\omega)| \leq C|\omega|^{-|\alpha|}.$$

Consequently, the dependence of the amplitude on μ does not interfere with the asymptotics, compare [6, Proposition 1.2.4]. Putting (3.8) and (3.11) together we obtain

$$\begin{aligned} K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) &= (\mu/2\pi)^{d-\frac{\text{codim Crit } \Phi_{\iota, x, y}}{2}} \frac{d_\sigma}{2\pi} \sum_\iota \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} e^{i\mu \Phi_{\iota, x, y}^0(R, t)} \left[\sum_{j=0}^{\tilde{N}-1} \mathcal{L}_\iota^j(R, t, x, y) \mu^{-j} \right. \\ &\quad \left. + \tilde{\mathcal{R}}_{\iota, \tilde{N}}(R, t, x, y, \mu) \right] dR dt \end{aligned}$$

up to terms of order $O(\mu^{-\infty})$ uniform in x and y . We now apply the stationary phase principle [13, Theorem 7.7.5] to the (R, t) -integral. If $y \in \mathcal{O}_x$, the phase function simply reads $t(1-R)$, and the only critical point is $(R_0, t_0) = (1, 0)$, which is non-degenerate, the determinant of the Hessian being -1 . Therefore, the necessary conditions for an application of the principle are fulfilled, yielding the assertion of the proposition in this case. In case that $y \notin \mathcal{O}_x$, the phase function is given by $t(1-R) + \Phi_{\iota, x, y}^0(R, t)$, and the determinant of the matrix of its second derivatives is given by

$$-(1 - c'_{x, y}(t))^2 \approx -(1 \pm O(\|\kappa_\iota(x) - \kappa_\iota(y \cdot k)\|))^2.$$

By choosing the charts Y_ι sufficiently small so that $\|\kappa_\iota(x) - \kappa_\iota(y \cdot k)\| \ll 1$, we can therefore achieve that in a sufficiently small neighborhood of $(R, t) = (1, 0)$, which is where the amplitude of the (R, t) -integral is supported, the phase function $t(1-R) + \Phi_{\iota, x, y}^0(R, t)$ has, if at all, only non-degenerate, hence isolated, critical points. If we now apply the stationary phase theorem, the proposition follows. \square

From Proposition 3.1 equivariant bounds for eigenfunctions can be easily inferred. Indeed, recall that the test function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$ was chosen such that $\varrho(0) = 1$ and $\text{supp } \hat{\varrho} \subset (-\delta/2, \delta/2)$ for some arbitrary $\delta > 0$. By choosing δ sufficiently small, one can even achieve that $\varrho > 0$ on $[-1, 1]$, compare [7, Proof of Lemma 2.3]. But then

$$\min_{\nu \in [-1, 1]} \varrho(\nu) \underbrace{\sum_{\substack{\mu_j \in (\mu, \mu+1], \phi_j \in L_\sigma^2(M) \\ = K_{s_\mu \circ \Pi_\sigma}(x, x)}} |\phi_j(x)|^2}_{=K_{s_\mu \circ \Pi_\sigma}(x, x)} \leq \underbrace{\sum_{\substack{j \geq 0, \phi_j \in L_\sigma^2(M) \\ = K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)}} \varrho(\mu - \mu_j) |\phi_j(x)|^2}_{=K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)} = O(d_\sigma \mu^{d-\kappa-1}),$$

so that $K_{s_\mu \circ \Pi_\sigma}(x, x) = O(d_\sigma \mu^{d-\kappa-1})$, compare [26, Remark 4.4 (2)]. In view of $\|s_\mu \circ \Pi_\sigma\|_{L^2 \rightarrow L^\infty}^2 \equiv \sup_{x \in M} K_{s_\mu \circ \Pi_\sigma}(x, x)$, one finally obtains the equivariant convex bound

$$\|\phi_j\|_\infty \ll d_\sigma \lambda_j^{\frac{d-\kappa-1}{2m}}$$

for any $\phi_j \in L_\sigma^2(M)$, see [26, Proposition 5.1].

4. SPECTRAL ASYMPTOTICS FOR KERNELS OF HECKE OPERATORS

The main goal of this paper consists in proving equivariant subconvex bounds for *Hecke–Maass forms*, that is, simultaneous eigenfunctions of a ring of Hecke operators and some invariant differential operator. To this purpose, we shall first derive asymptotics for kernels of Hecke operators in the eigenvalue aspect.

4.1. Equivariant kernel asymptotics. Keep the notation of Sections 2 and 3, and consider the compact d -dimensional Riemannian manifold $M = \Gamma_\chi \backslash G$. Note that K acts on G and on M from the right in an isometric and effective way, the isotropy group of a point $\Gamma_\chi g \in M$ being conjugate to the finite group $gKg^{-1} \cap \Gamma_\chi$. Hence, all K -orbits in M are either principal or exceptional, and of dimension $\dim K$. Since the maximal compact subgroups of G are precisely the conjugates of K , exceptional K -orbits arise from elements in Γ_χ of finite order. Consider now the right regular representation π of K on $L^2(M)$ together with the corresponding Peter-Weyl decomposition (3.5), and suppose that P commutes with π and the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$, which commute with the right regular K -representation as well. To describe the growth of simultaneous eigenfunctions of P and $T_{\Gamma\beta\Gamma}^\chi$ in the σ -isotypic component

$$(4.1) \quad L_{\sigma, \chi}^2(\Gamma \backslash G) := L_\sigma^2(\Gamma_\chi \backslash G) \cap L_\chi^2(\Gamma \backslash G), \quad \chi \in \widehat{\Gamma}, \sigma \in \widehat{K},$$

of $L_\chi^2(\Gamma \backslash G)$, we are interested in spectral asymptotics for the Schwartz kernel of the operator

$$\Pi_\sigma \circ T_{h(\chi)} \circ \mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ Q \circ T_{h(\chi)} \circ \Pi_\sigma = \mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ Q \circ \Pi_\sigma : L^2(\Gamma_\chi \backslash G) \rightarrow L^2(\Gamma_\chi \backslash G).$$

Applying the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ to the spectral expansion (3.6) of the spectral function of $Q \circ \Pi_\sigma$ yields

$$(4.2) \quad \sum_{\substack{\mu_j \leq \mu, \phi_j \in L_\sigma^2(M)}} \lambda_j(\chi, \beta) \phi_j(x) \overline{\phi_j(y)} = [\Gamma : \Gamma_\chi]^{-1} \sum_{\substack{\mu_j \leq \mu, \phi_j \in L_\sigma^2(M) \\ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma}} \overline{\chi(\alpha)} \phi_j(\alpha \cdot x) \overline{\phi_j(y)},$$

where we wrote $\lambda_j(\chi, \beta) \phi_j$ for the eigenvalue of $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ belonging to the Hecke–Maass form ϕ_j , compare [16, Eq. (1.4)]. In order to get an asymptotic description of the right-hand side of (4.2), we consider the composition $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ s_\mu \circ \Pi_\sigma$ together with the corresponding composition $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma$ with the approximate spectral projection \tilde{s}_μ . Clearly, its Schwartz kernel can be written as

$$(4.3) \quad K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, y) := \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, y),$$

$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ being as in (3.7), and by (3.10) one immediately deduces

$$K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) = O(d_\sigma |\Gamma_\chi \backslash \Gamma\beta\Gamma| \mu^{d-\dim K-1})$$

uniformly in x . Nevertheless, to obtain subconvex bounds, more subtle estimates are necessary.

Proposition 4.1. *Let $\chi \in \widehat{\Gamma}$, $x \in M = \Gamma_\chi \backslash G$, and $\tilde{N} \in \mathbb{N}$ be fixed. Assume that the cospheres $S_x^* M := \{(x, \xi) \in T^* M \mid p(x, \xi) = 1\}$ are strictly convex. Then, up to terms of order $O(\mu^{-\infty})$, one has the asymptotic expansion*

$$\begin{aligned} K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &= \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \alpha x \in xK}} \left[\sum_{j=0}^{\tilde{N}} \mathcal{L}_j(x, \alpha) \mu^{d-\dim K-1-j} + O_{x,\alpha}(\mu^{d-\dim K-1-\tilde{N}-1}) \right] \\ &+ \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \alpha x \notin xK}} \left[\sum_{j=0}^{\tilde{N}} \mathcal{M}_j(x, \alpha) \mu^{(d-\dim K-1)/2-j} + \mathcal{R}_{\tilde{N}}(x, \alpha, \mu) \right] \end{aligned}$$

as $\mu \rightarrow +\infty$, with known coefficients $\mathcal{L}_j(x, \alpha)$, $\mathcal{M}_j(x, \alpha)$ and remainders. Furthermore, $\mathcal{L}_j(x, \alpha)$ and $O_{x,\alpha}(\mu^{d-\dim K-1-\tilde{N}-1})$ are uniformly bounded in x and α , while

$$\begin{aligned} |\mathcal{M}_j(x, \alpha)| &\leq C_j \operatorname{dist}(x, \alpha x K)^{-(d-1)/2-j(d-1)}, \\ |\mathcal{R}_{\tilde{N}}(x, \alpha, \mu)| &\leq C_{\tilde{N}} \operatorname{dist}(x, \alpha x K)^{-(d-1)/2-\tilde{N}(d-1)-1} \mu^{(d-\dim K-1)/2-\tilde{N}-1} \end{aligned}$$

for some constants $C_j, C_{\tilde{N}} > 0$ independent of μ, x , and α .

Proof. This follows directly from (4.3), Proposition 3.1, and the fact that $|\chi(\alpha)| = 1$. Note that the condition $\alpha x \in xK$ in the first sum over α implies that $\operatorname{dist}(x, \alpha x K) = 0$. \square

As a consequence, we obtain

Corollary 4.2. *In the situation of Proposition 4.1 we have*

$$\begin{aligned} K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &\ll \mu^{d-\dim K-1} M_1(x, \beta) + \mu^{(d-\dim K-1)/2} \int_0^C s^{-1/2} dM_2(s) \\ &+ \mu^{(d-\dim K-1)/2-1} \int_0^C s^{-1/2-1/(d-1)} dM_2(s), \end{aligned}$$

for some constant $\operatorname{diam} \Gamma_\chi \backslash G / K < C < \infty$, where for $\delta > 0$ we set

$$\begin{aligned} M_1(x, \beta) &:= \# \{ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma : \alpha x \in xK \}, \\ M_2(\delta) &:= M_2(x, \beta, \delta) := \# \{ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma : \alpha x \notin xK, \operatorname{dist}(x, \alpha x K)^{d-1} < \delta \}. \end{aligned}$$

Proof. By definition of the Stieltjes integral we have

$$\sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \alpha x \notin xK} \operatorname{dist}(x, \alpha x K)^a = \int_0^C s^{a/(d-1)} dM_2(s), \quad a \in \mathbb{R},$$

and the assertion follows from the previous proposition by taking $\tilde{N} = 0$. \square

The estimate in Corollary 4.2 is quite sharp, but the behavior of $M_2(x, \beta, \delta)$ is difficult to control for small δ . For this reason, we shall rely on the following less sharp estimate, which nevertheless will be sufficient to derive equivariant subconvex bounds.

Corollary 4.3. *Let $\delta > 0$. Then, in the situation of Proposition 4.1 one has*

$$\begin{aligned} K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &\ll \mu^{d-\dim K-1} M(x, \beta, \delta) + \mu^{(d-\dim K-1)/2} \int_\delta^C s^{-1/2} dM(s) \\ &+ \mu^{(d-\dim K-1)/2-1} \int_\delta^C s^{-1/2-1/(d-1)} dM(s) \end{aligned}$$

uniformly in $x \in \Gamma_\chi \backslash G$ for some constant $\operatorname{diam} \Gamma_\chi \backslash G / K < C < \infty$, where we set

$$(4.4) \quad M(\delta) := M(x, \beta, \delta) := \# \{ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma : \operatorname{dist}(xK, \alpha x K)^{d-1} < \delta \}.$$

Proof. By Proposition 3.1 one deduces for $\delta^{-1} \leq \mu$

$$\begin{aligned} & \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha xK)^{d-1} \geq \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ & \ll \mu^{(d-\dim K-1)/2} \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha xK)^{d-1} \geq \delta}} \left(\text{dist}(xK, \alpha xK)^{-(d-1)/2} + \text{dist}(xK, \alpha xK)^{-(d-1)/2-1} \mu^{-1} \right). \end{aligned}$$

Furthermore, by (3.10) one has the uniform bound $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = O(\mu^{d-\dim K-1})$. In view of (4.3) we therefore obtain

$$\begin{aligned} [\Gamma : \Gamma_\chi] K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &= \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha xK)^{d-1} < \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ &+ \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha xK)^{d-1} \geq \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ &\ll \mu^{d-\dim K-1} M(x, \beta, \delta) + \mu^{(d-\dim K-1)/2} \int_\delta^C s^{-1/2} dM(s) \\ &+ \mu^{(d-\dim K-1)/2-1} \int_\delta^C s^{-1/2-1(d-1)} dM(s) \end{aligned}$$

by definition of the Stieltjes integral, and the assertion follows. \square

4.2. Non-equivariant kernel asymptotics. In what follows, we shall apply our approach also to non-equivariant situations. Keeping the notation of Sections 2 and 3, we shall thus examine the asymptotic behavior of the Schwartz kernel of $\sqrt[p]{P} \circ \mathcal{T}_{\Gamma\beta\Gamma}^\chi$ by means of Fourier integral operator methods. Though the spherical subconvex bounds we shall deduce from this are essentially known, the kernel asymptotics derived in this section might be useful in proving non-equivariant subconvex bounds for Hecke–Maass forms on $\Gamma_\chi \backslash G$.

Consider the compact Riemannian manifold $M = \Gamma_\chi \backslash X$, where $X = G$ or $X = G/K$, and write $d = \dim X$. Assume that P commutes with the operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$. In order to study the growth of simultaneous eigenfunctions of P and $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$, we are interested in spectral asymptotics for the Schwartz kernel of the operator

$$T_{h(\chi)} \circ T_{\Gamma\beta\Gamma}^\chi \circ Q \circ T_{h(\chi)} = \mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ Q : L^2(M) \rightarrow L^2(M).$$

An application of $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ to the spectral expansion (3.1) of $e(x, y, \mu)$ yields

$$\sum_{\mu_j \leq \mu} \lambda_j(\chi, \beta) \phi_j(x) \overline{\phi_j(y)} = [\Gamma : \Gamma_\chi]^{-1} \sum_{\mu_j \leq \mu, \alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma} \overline{\chi(\alpha)} \phi_j(\alpha \cdot x) \overline{\phi_j(y)}.$$

Now, consider the composition $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ s_\mu$ or rather $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu$, the kernel of the latter being given by

$$K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu}(x, y) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma} \overline{\chi(\alpha)} K_{\tilde{s}_\mu}(\alpha \cdot x, y) \in C^\infty(M \times M).$$

Approximating \tilde{s}_μ by means of Fourier integral operators as in Section 3 one obtains for the kernel the expression

$$(4.5) \quad K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu}(x, y) = \frac{\mu^d}{(2\pi)^{d+1} [\Gamma : \Gamma_\chi]} \sum_{\iota} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma} \overline{\chi(\alpha)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} I_\iota(\mu, R, t, x, y, \alpha) dR dt$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y , where

$$(4.6) \quad I_\iota(\mu, R, t, x, y, \alpha) := \int_{\Sigma_{\iota, x}^{R, t}} e^{i\mu\Phi_{\iota, x, y, \alpha}(\omega)} \hat{g}(t) f_\iota(x) \cdot a_\iota(t, \kappa_\iota(x), \mu\omega) \bar{f}_\iota(\alpha \cdot y) b(q(x, \omega)) J_\iota(\alpha, y) d\Sigma_{\iota, x}^{R, t}(\omega),$$

and $\Phi_{\iota, x, y, \alpha}(\omega) := \langle \kappa_\iota(x) - \kappa_\iota(\alpha \cdot y), \omega \rangle$, the other notation being as in (3.8). To obtain an asymptotic description as $\mu \rightarrow +\infty$, we shall first apply the stationary phase theorem to the integrals $I_\iota(\mu, R, t, x, y, \alpha)$, and then to the integral over R and t . Thus, we are interested in the asymptotic behavior of oscillatory integrals of the form

$$(4.7) \quad I_z(\nu) := \int_{\Sigma_x^{R, t}} e^{i\nu\Psi_z(\omega)} a(\omega) d\Sigma_x^{R, t}(\omega), \quad z \in S^{n-1}, \quad \nu \rightarrow +\infty,$$

with $\Sigma_x^{R, t}$ as in (3.9) and phase function $\Psi_z(\omega) := \langle z, \omega \rangle$, while $a \in C_c^\infty$ is an amplitude that might depend smoothly on ν and some other parameters. Here we skipped the index ι for simplicity of notation. Performing stationary phase one obtains

Proposition 4.4. *Assume that the cospheres S_x^*M are strictly convex and $t \ll 1$. Then, for every $L \in \mathbb{N}$ one has the asymptotic formula*

$$I_z(\nu) = \sum_{\omega_0 \in \text{Crit } \Psi_z} \frac{e^{i\nu\Psi_z(\omega_0)}}{(\det(\nu\Pi_{\omega_0}/2\pi i))^{1/2}} \left[\sum_{j=0}^L Q_j(a, \omega_0) \nu^{-j} + \mathcal{R}_L(\nu) \right]$$

as $\mu \rightarrow +\infty$, where the critical set of the phase function Ψ_z is given by

$$\text{Crit } \Psi_z := \left\{ \omega \in \Sigma_x^{R, t} \mid z \in N_\omega \Sigma_x^{R, t} \right\},$$

and consists only consists of non-degenerate, isolated points, while Π denotes the second fundamental form of $\Sigma_x^{R, t}$. The coefficients $Q_j(a, \omega_0)$ and the remainder $\mathcal{R}_L(\nu)$ satisfy the bounds

$$|Q_j| \leq C_j \sup_{l \leq 2j} |D^l a(\omega_0)|, \quad |\mathcal{R}_L(\nu)| \leq C_L \sup_{l \leq d-1+2L+2} \|D^l a\|_{\infty, \Sigma_x^{R, t}} \mu^{-L-1}$$

for suitable constants $C_j, C_L > 0$ independent of z and ν , where D^l denotes a differential operator on $\Sigma_x^{R, t}$ of order l . Furthermore,

$$\Psi_z(\omega_0) = R c_{x, \omega_0}(t), \quad c_{x, \omega_0}(t) := \pm \|\text{grad}_\eta \zeta(t, \kappa(x), \omega_0)\|^{-1}.$$

Proof. The statement of the proposition is essentially known [13, Theorem 7.7.14], but for the convenience of the reader, we include a proof here. Consider a local parametrization

$$(4.8) \quad F: \mathbb{R}^{d-1} \supset U \longrightarrow \Sigma_x^{R, t} \subset \mathbb{R}^d, \quad \xi \longmapsto F(\xi) = \omega,$$

of the hypersurface $\Sigma_x^{R, t}$. If we differentiate Ψ_z with respect to the ξ -coordinates we arrive at the conditions $\langle z, \partial F / \partial \xi_i \rangle = 0$ for $i = 1, \dots, n-1$, implying that z must be normal to $\Sigma_x^{R, t}$ at ω . Thus, $\text{Crit } \Psi_z = \{\omega \in \Sigma_x^{R, t} : z \in N_\omega \Sigma_x^{R, t}\}$. Now, assume that the cospheres S_x^*M are strictly convex. For small $|t| \ll 1$, the hypersurfaces $\Sigma_x^{R, t}$ will be strictly convex, too. In particular, $\Sigma_x^{R, t}$ is orientable, and the Gauss map

$$\mathcal{N}: \Sigma_x^{R, t} \ni \omega \longmapsto \mathcal{N}(\omega) \in N_\omega \Sigma_x^{R, t},$$

which assigns to each point of $\Sigma_x^{R, t}$ the *outer* normal unit vector to $\Sigma_x^{R, t}$ at that point, is a global diffeomorphism. Therefore, for each $\tilde{z} \in S^{n-1}$ there is a unique $\omega_{\tilde{z}} \in \Sigma_x^{R, t}$ such that $\tilde{z} = \mathcal{N}(\omega_{\tilde{z}})$. Consequently, $\omega \in \text{Crit } \Psi_z$ is locally uniquely determined by the condition $\mathcal{N}(\omega) = \pm \mathcal{N}(\omega_z)$, so that ω is an isolated point. In order to see that $\text{Crit } \Psi_z$ consists of non-degenerate points, note that with respect to the parametrization (4.8) of $\Sigma_x^{R, t}$ the Hessian of Ψ_z at a critical point ω is given by the matrix

$$(4.9) \quad \text{Hess } \Psi_z(\omega) \equiv \left(\left\langle z, \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} (F^{-1}(\omega)) \right\rangle \right)_{1 \leq i, j \leq d-1}.$$

Since $z \in N_\omega \Sigma_x^{R,t}$, $\text{Hess } \Psi_z(\omega)$ essentially corresponds to the second fundamental of $\Sigma_x^{R,t}$

$$\Pi : T\Sigma_x^{R,t} \times T\Sigma_x^{R,t} \longrightarrow C^\infty(\Sigma_x^{R,t}), \quad \Pi(\mathcal{X}, \mathcal{Y}) := \langle \nabla_{\mathcal{X}} \mathcal{Y}, \mathcal{N} \rangle = \langle \mathcal{X}, A \mathcal{Y} \rangle,$$

where $\nabla_{\mathcal{X}} \mathcal{Y} \equiv \mathcal{X}(\mathcal{Y})$ denotes the covariant derivative in Euclidean space \mathbb{R}^d , and $A : T\Sigma_x^{R,t} \rightarrow T\Sigma_x^{R,t}$ the symmetric endomorphism induced by Π [19, Chapter VII, Section 3]. Indeed, assume that $z = -\mathcal{N}(\omega)$, and let $\partial/\partial \xi_{i|\omega} := \partial F(\xi^{-1}(\omega))/\partial \xi_i$, $1 \leq i \leq n-1$, be the coordinate frame given by the parametrization (4.8). Then, the matrix in (4.9) reads

$$(4.10) \quad -\Pi \left(\frac{\partial}{\partial \xi_{i|\omega}}, \frac{\partial}{\partial \xi_{j|\omega}} \right) = - \left\langle \frac{\partial}{\partial \xi_{i|\omega}}, A \frac{\partial}{\partial \xi_{j|\omega}} \right\rangle.$$

Since $\Sigma_x^{R,t}$ is strictly convex, the eigenvalues of the second fundamental form of $\Sigma_x^{R,t}$ at ω , which are given by the principal curvatures of $\Sigma_x^{R,t}$ at that point, are all non-zero. Therefore, the determinant of $\text{Hess } \Psi_z(\omega)$ is non-zero, and ω must be a non-degenerate critical point. In conclusion, Ψ_z has a clean critical set, so that the asymptotic formula for $I_z(\nu)$ follows directly by applying the stationary phase theorem [13, Theorem 7.7.5] to $I_z(\nu)$. In particular,

$$Q_j = \left[\sum_{r-k=j} \sum_{3k \leq 2r} \frac{1}{r! k! 2^r i^{r-k}} \left(\langle D_\xi, \Pi_{\omega_0}^{-1} D_\xi \rangle^r (H^k a) \right) (\omega_0) \right],$$

where $D_\xi \equiv (\partial/\partial \xi_1, \dots, \partial/\partial \xi_{d-1})$ is given in terms of (4.8), and $H(\omega)$ is a smooth function of Ψ_z which vanishes of third order at ω_0 , so that in the expression for Q_j we must have $r \leq 3j$ and $k \leq 2j$, and Q_j corresponds to a differential operator of order $2j$, yielding the estimate for Q_j . The estimate for the remainder follows from [25, Eq. (A.2)] if one sets $N = L + (d-1)/2$ there⁵. Finally, regarding the value of Ψ_z on its critical set, note that for $\omega_0 \in \text{Crit } \Psi_z$ one computes

$$\Psi_z(\omega_0) = \langle z, \omega_0 \rangle = \pm c_{x, \omega_0}(t) \langle \text{grad}_\eta \zeta(t, \kappa(x), \omega_0), \omega_0 \rangle = \pm R c_{x, \omega_0}(t),$$

since z must be colinear to $\text{grad}_\eta \zeta(t, \kappa(x), \omega_0)$. In particular notice that $c_{x, \omega_0}(t)$ is independent of R due to the fact that $\zeta(t, \kappa(x), \eta)$ is homogeneous of degree 1 in η , so that $\text{grad}_\eta \zeta(t, \kappa(x), \omega_0)$ only depends on the direction of ω_0 . \square

Let us now come back to our initial question of finding asymptotics for the kernels $K_{T_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu}(x, x)$.

Proposition 4.5. *Let $X = G$ or G/K , and $d = \dim X$. Further, assume that the cospheres $S_x^* M$ are strictly convex and let $\chi \in \hat{\Gamma}$, $x \in \Gamma_\chi \backslash X$, and $\tilde{N} \in \mathbb{N}$ be fixed. Then, as $\mu \rightarrow +\infty$, one has the asymptotic expansion*

$$(4.11) \quad K_{T_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu}(x, x) = \left[\sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \text{dist}(x, \alpha \cdot x) < \mu^{-1}}} \left[\sum_{k=0}^{\tilde{N}} \mathcal{L}_k(x, \alpha) \mu^{d-1-k} + O_{x, \alpha}(\mu^{d-1-\tilde{N}-1}) \right] \right. \\ \left. + \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \text{dist}(x, \alpha \cdot x) \geq \mu^{-1}}} \left[\sum_{k=0}^{\tilde{N}} \mathcal{M}_k(x, \alpha) \mu^{(d-1)/2-k} + \mathcal{R}_{\tilde{N}}(x, \alpha, \mu) \right] \right]$$

up to terms of order $O(\mu^{-\infty})$, with known coefficients $\mathcal{L}_k(x, \alpha)$, $\mathcal{M}_k(x, \alpha)$ and remainder estimates that depend smoothly on x and α . Furthermore, $\mathcal{L}_k(x, \alpha)$ and $O_{x, \alpha}(\mu^{d-1-\tilde{N}-1})$ are uniformly bounded in x and α , while

$$(4.12) \quad |\mathcal{M}_k(x, \alpha)| \leq C_k \text{dist}(x, \alpha \cdot x)^{-(d-1)/2-k}, \\ |\mathcal{R}_{\tilde{N}}(x, \alpha, \mu)| \leq C_{\tilde{N}} \text{dist}(x, \alpha \cdot x)^{-(d-1)/2-\tilde{N}-1} \mu^{(d-1)/2-\tilde{N}-1}$$

for some constants $C_k, C_{\tilde{N}} > 0$ independent of μ, x , and α .

⁵Note that there should be no square brackets in [25, Eq. (A.2)].

Proof. Let $\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma$ be such that $\alpha \cdot y \neq x$. Applying Proposition 4.4 to the integral (4.6) with $\left\langle \frac{\kappa_\ell(x) - \kappa_\ell(\alpha \cdot y)}{\|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot y)\|}, \omega \right\rangle$ as phase function and $\nu := \mu \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot y)\|$ as a parameter yields for any $L \in \mathbb{N}$ the expansion

$$I_\ell(\mu, R, t, x, y, \alpha) = \sum_{\omega_0 \in \text{Crit}\Phi_{\ell, x, y, \alpha}} e^{i\mu\Phi_{\ell, x, y, \alpha}(\omega_0)} \left[\sum_{j=0}^L \tilde{Q}_{\ell, j} \mu^{-j-(d-1)/2} + \tilde{\mathcal{R}}_{\ell, L}(\mu) \right],$$

where the coefficients and the remainder are smooth in all parameters and satisfy the bounds

$$|\tilde{Q}_{\ell, j}| \leq \tilde{C}_j \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot y)\|^{-(d-1)/2-j},$$

$$|\tilde{\mathcal{R}}_{\ell, L}(\mu)| \leq \tilde{C}_L (\|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot y)\| \mu)^{-(d-1)/2-L-1}$$

for suitable constants $\tilde{C}_j, \tilde{C}_L > 0$, while $\Phi_{\ell, x, y, \alpha}(\omega_0) = \pm \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot x)\| R / \|\text{grad}_\eta \zeta_\ell(t, \kappa(x), \omega_0)\|$. From this and (4.5) we deduce for $\mu \geq 1$

$$K_{T_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu}(x, x) = \frac{\mu^d}{(2\pi)^{d+1} [\Gamma : \Gamma_\chi]} \sum_{\iota} \left[\sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot x)\| < \mu^{-1}}} \overline{\chi(\alpha)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} I_\ell(\mu, R, t, x, x, \alpha) dR dt \right. \\ \left. + \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma, \\ \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot x)\| \geq \mu^{-1}, \\ \omega_0 \in \text{Crit}\Phi_{\ell, x, x, \alpha}}} \overline{\chi(\alpha)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt+\Phi_{\ell, x, x, \alpha}(\omega_0)]} \left[\sum_{j=0}^L \tilde{Q}_{\ell, j} \mu^{-j-(d-1)/2} + \tilde{\mathcal{R}}_{\ell, L}(\mu) \right] dR dt \right]$$

up to terms of order $O(\mu^{-\infty})$. We now apply the stationary phase principle [13, Theorem 7.7.5] to the (R, t) -integrals as in the proof of Proposition 3.1. For the first group of integrals one has to differentiate the integrals $I_\ell(\mu, R, t, x, y, \alpha)$ and, in particular, the phase $e^{i\mu\Phi_{\ell, x, y, \alpha}(\omega)}$ with respect to (R, t) , which, nevertheless, will just yield contributions of order $\mu \|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot x)\| < 1$, by the choice of α . For the second group of integrals no such contributions arise. Taking into account that $|\chi(\alpha)| = 1$ for all α we obtain the assertion. \square

Remark 4.6. (1) Note that in the previous theorem no stationary phase was applied to the integrals $I_\ell(\mu, R, t, x, x, \alpha)$ in case that $\|\kappa_\ell(x) - \kappa_\ell(\alpha \cdot x)\| < \mu^{-1}$. This will allow us to control the critical contributions coming from points x that are close to fixed points of elliptic elements α , and at the end obtain uniform estimates in x .
(2) In (4.12), it is crucial not to estimate $\mathcal{M}_k(x, \alpha)$ and $\mathcal{R}_{\tilde{N}}(x, \alpha, \mu)$ simply by $O(\mu^{(d-1)/2+k})$ and $O(\mu^{d-1})$, respectively, which would destroy the possibility of any subconvex bound.

5. EQUIVARIANT SUBCONVEX BOUNDS FOR ARITHMETIC CONGRUENCE LATTICES IN $\text{SL}(2, \mathbb{R})$

In this section, we shall use the kernel asymptotics derived in the previous section to prove spherical and non-spherical subconvex bounds for arithmetic congruence lattices in the setting considered by Iwaniec and Sarnak [16].

5.1. Congruence arithmetic lattices. Let A be an indefinite quaternion division algebra over \mathbb{Q} . Hence, there exist two square-free integers a and b such that $a > 0$ and

$$A = \mathbb{Q} + \mathbb{Q}\omega + \mathbb{Q}\Omega + \mathbb{Q}\omega\Omega$$

where $\omega^2 = a$, $\Omega^2 = b$, and $\omega\Omega = -\Omega\omega$. For each element $x = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega$, its conjugate is defined as $\bar{x} := x_0 - x_1\omega - x_2\Omega - x_3\omega\Omega$, and its trace and norm as $\text{tr}(x) := x + \bar{x}$ and $N(x) := x\bar{x}$, respectively. Let \mathcal{R} be an order of A , that is, \mathcal{R} is a finitely generated free \mathbb{Z} -module, \mathcal{R} is a subring of A containing 1, and $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. For each prime number p , set $A_p := A \otimes \mathbb{Q}_p$ and $\mathcal{R}_p := \mathcal{R} \otimes \mathbb{Z}_p$. Let d_A be the product of all primes p such that A_p is a division algebra. Then d_A is called the *discriminant* of A . d_A is greater than 1 and square free, and A_p is isomorphic to $M(2, \mathbb{Q}_p)$ if p does not divide q .

Throughout this section, we assume that \mathcal{R} is an *Eichler order of level L* , where L is a natural number such that $(d_A, L) = 1$. Hence, \mathcal{R} satisfies

- (1) \mathcal{R}_p is the maximal order of A_p if p divides d_A , or
- (2) \mathcal{R}_p is conjugate to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ L\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$.

Note that any Eichler order is included in a maximal order. Particularly, \mathcal{R} is maximal when $L = 1$. Now, choose an embedding $\theta : A \rightarrow M(2, \mathbb{Q}(\sqrt{a})) \subset M(2, \mathbb{R})$ by setting

$$\theta(x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega) = \begin{pmatrix} x_0 - x_1\omega & x_2 + x_3\omega \\ b(x_2 - x_3\omega) & x_0 + x_1\omega \end{pmatrix}.$$

For each natural number n , we set

$$\mathcal{R}(n) := \{\alpha \in \mathcal{R} \mid N(\alpha) = n\}.$$

Then $\Gamma := \theta(\mathcal{R}(1))$ becomes a cocompact lattice of $G := \mathrm{SL}(2, \mathbb{R})$. Note that $\mathrm{tr}(x) = \mathrm{tr}(\theta(x))$ and $N(x) = \det(\theta(x))$ hold for any x in A . In what follows, we identify A with $\theta(A)$. Especially, we will often use Γ instead of $\mathcal{R}(1)$.

Next, let χ be a Dirichlet character on $(\mathbb{Z}/L\mathbb{Z})^\times$. In view of the product isomorphism $(\mathbb{Z}/L\mathbb{Z})^\times \cong \prod_{p|L} (\mathbb{Z}_p/L\mathbb{Z}_p)^\times$ given by the diagonal embedding $a \mapsto (a)_p$, a character χ_p can be defined on $(\mathbb{Z}_p/L\mathbb{Z}_p)^\times$ by restriction of χ to each factor. Set

$$\Xi_{\mathcal{R}} := \{\alpha \in \mathcal{R} \mid N(\alpha) > 0\} \quad \text{and} \quad \mathcal{R}_L := \{(x_p)_{p|L} \mid x_p \in \mathcal{R}_p, \ N(x_p) \neq 0\},$$

and define a character χ_L on the semigroup \mathcal{R}_L by

$$\mathcal{R}_L \ni \left(\begin{pmatrix} a_p & b_p \\ Lc_p & d_p \end{pmatrix} \right)_{p|L} \mapsto \prod_{p|L} \overline{\chi_p(a_p)} \in \mathbb{C}^1.$$

Composing χ_L and the diagonal embedding $\Xi_{\mathcal{R}} \subset \mathcal{R}_L$, we obtain a character χ on the sub-semigroup $\Xi_{\mathcal{R}}$ of A^\times . By the inclusion $\Gamma \subset \Xi_{\mathcal{R}}$, χ becomes a character on Γ which is called a *Nebentypus character*.

Now, notice that because Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable [24, Proposition 4.1], an inclusion map $\psi : \mathcal{R}(n) \rightarrow C(\Gamma)$ is given by

$$\psi(\alpha) = n^{1/2}\alpha, \quad \alpha \in \mathcal{R}(n).$$

Since the subset $\mathcal{R}(n)$ is left and right Γ -invariant, and it is known that $\Gamma \backslash \mathcal{R}(n)$ is finite [23, Section 5.3], we can introduce the Hecke operators

$$(T_n^\chi f)(x) := \sum_{\alpha \in \Gamma \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} f(\alpha \cdot x), \quad f \in L_\chi^2(\Gamma \backslash G),$$

where we wrote $\alpha \equiv \Gamma\alpha$ for short. Indeed, T_n^χ coincides with the Hecke operators $T_{\psi(\mathcal{R}(n))}^\chi = \sum_j T_{\Gamma\alpha_j\Gamma}^\chi$ defined in (2.7), where $\psi(\mathcal{R}(n)) = \sqcup_j \Gamma\alpha_j\Gamma$. In particular, we have also the Hecke operators

$$\mathcal{T}_n^\chi : L^2(\Gamma_\chi \backslash G) \longrightarrow L^2(\Gamma_\chi \backslash G), \quad (\mathcal{T}_n^\chi f)(x) := \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

compare (2.2) and Diagram 2.10. We now set

$$q := d_A L.$$

For natural numbers n such that $(n, q) = 1$, the T_n^χ are self-dual, commute with the Laplace-Beltrami operator Δ on G , and satisfy the composition rule [23, Section 5.3]

$$(5.1) \quad T_n^\chi T_m^\chi = \sum_{d|(m,n)} d \cdot \chi(d) T_{nm/d^2}^\chi.$$

Next, let $K := \mathrm{SO}(2)$, $\sigma \in \widehat{K}$ be a fixed K -type, and $L^2_{\sigma, \chi}(\Gamma \backslash G)$ be defined as in (4.1). Further, let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(\Gamma_\chi \backslash G)$ consisting of simultaneous eigenfunctions of $P = \Delta$ and \mathcal{T}_n^χ compatible with the decompositions (2.6) and (3.5) so that with $(n, q) = 1$

$$(5.2) \quad \Delta \phi_j + \lambda_j = 0, \quad \mathcal{T}_n^\chi \phi_j = \lambda_j(n) \phi_j.$$

Note that $\lambda_j(n) = 0$ if $\phi_j \notin L^2_\chi(\Gamma \backslash G)$. When σ is trivial, the space $L^2_{\sigma, \chi}(\Gamma \backslash G)$ can be identified with $L^2_\chi(\Gamma \backslash G/K)$, in which case we omit σ in the above notations.

5.2. The spherical case. In what follows, we shall examine first the spherical case, and consider for this the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$. The group $\mathrm{GL}(2, \mathbb{R})^+ := \{x \in M(2, \mathbb{R}) \mid \det(x) > 0\}$ acts transitively on \mathbb{H} by fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)(cz + d)^{-1}, \quad z \in \mathbb{H},$$

and $\mathbb{H} \simeq G/K$, where $K = \mathrm{SO}(2)$. Note that the center $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{R}^*$, acts trivially on \mathbb{H} . In what follows, we identify $M := \Gamma_\chi \backslash G/K \simeq \Gamma_\chi \backslash \mathbb{H}$ with a subset in \mathbb{H} , and endow it with the standard hyperbolic metric on \mathbb{H} given by

$$\mathrm{dist}(z, w) := \mathrm{arcosh}(1 + u(z, w)/2) = \ln(1 + u(z, w)/2 + \sqrt{(1 + u(z, w)/2)^2 - 1}),$$

where

$$(5.3) \quad u(z, w) := \frac{|z - w|^2}{\mathrm{Im} z \mathrm{Im} w}, \quad z, w \in \mathbb{H}.$$

By this, M becomes a compact hyperbolic surface. Our next goal is to derive a uniform bound for

$$K_{\mathcal{T}_n^\chi \circ \widetilde{s}_\mu}(z, w) := \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{j \geq 0, \alpha \in \Gamma_\chi \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} \varrho(\mu - \mu_j) \phi_j(\alpha \cdot z) \overline{\phi_j(w)}$$

on the diagonal in terms of μ and n , the notation being as in Section 3. In order to do so, choose a pre-compact open subset $M \subset Y \subset \mathbb{H}$, and let κ be the identity map on Y . We can then take (κ, Y) as coordinates on M . In particular, for suitable constants $c_1, c_2 > 0$

$$(5.4) \quad c_1 u(z, w) \leq \|\kappa(z) - \kappa(w)\|^2 = |z - w|^2 \leq c_2 u(z, w), \quad z, w \in \Gamma_\chi \backslash \mathbb{H}.$$

Adapting Proposition 4.5 to the present context yields

$$(5.5) \quad K_{\mathcal{T}_n^\chi \circ \widetilde{s}_\mu}(z, z) \ll \mu M(z, n, \mu^{-2}) + \sum_{\substack{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n) \\ u(z, \alpha \cdot z) \geq \mu^{-2}}} \left[\mu^{1/2} u(z, \alpha \cdot z)^{-1/4} + \underbrace{u(z, \alpha \cdot z)^{-3/4} \mu^{-1/2}}_{\leq \mu^{1/2} u(z, \alpha \cdot z)^{-1/4}} \right],$$

where we took into account (5.4), and introduced the lattice point counting function

$$M(\delta) := M(z, n, \delta) := \#\{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n) : u(z, \alpha \cdot z) < \delta\}, \quad \delta > 0.$$

Note that since $\Gamma_\chi \backslash \mathcal{R}(n)$ is finite, $M(z, n, \delta) \leq |\Gamma_\chi \backslash \mathcal{R}(n)|$ is a bounded function in δ . Now, we have the following

Lemma 5.1. *For arbitrary $\varepsilon > 0$ one has*

$$M(z, n, \delta) \leq \#\{\alpha \in \mathcal{R}(n) : u(z, \alpha \cdot z) < \delta\} \ll_\varepsilon (\delta + \delta^{1/4}) n^{1+\varepsilon} + n^\varepsilon$$

uniformly in z .

Proof. See [16, Lemma 1.3]. □

Now,

$$\sum_{\substack{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n) \\ u(z, \alpha \cdot z) \geq \delta}} u(z, \alpha \cdot z)^{-1/4} = \int_\delta^C s^{-1/4} dM(s)$$

for some sufficiently large constant $\text{diam } \Gamma_\chi \backslash \mathbb{H} < C < \infty$ independent of n . Furthermore, partial integration for Stieltjes integrals yields with $N(s) = s^{-1/4}$ and the previous lemma

$$\begin{aligned} \int_\delta^C s^{-1/4} dM(s) &= N(C)M(C) - N(\delta)M(\delta) - \int_\delta^C M(s) \underbrace{dN(s)}_{=N'(s) ds} \\ &\ll_\varepsilon n^{1+\varepsilon} + \delta^{-1/4}[(\delta + \delta^{1/4})n^{1+\varepsilon} + n^\varepsilon] + \underbrace{\int_\delta^C [(s + s^{1/4})n^{1+\varepsilon} + n^\varepsilon] s^{-5/4} ds}_{= \left(\frac{4}{3} s^{3/4} + \log s \right) n^{1+\varepsilon} - 4s^{-1/4} n^\varepsilon \Big|_\delta^C} \\ &= \left(\frac{4}{3} s^{3/4} + \log s \right) n^{1+\varepsilon} - 4s^{-1/4} n^\varepsilon \Big|_\delta^C. \end{aligned}$$

Taking everything together we obtain for $\mu \geq 1$ and $\delta = \mu^{-2}$ the uniform bound

$$\begin{aligned} (5.6) \quad K_{\mathcal{T}_n^\chi \circ \tilde{s}_\mu}(z, z) &\ll_\varepsilon \mu [(\mu^{-2} + \mu^{-1/2})n^{1+\varepsilon} + n^\varepsilon] \\ &+ \mu^{1/2} \left[n^{1+\varepsilon} + \mu^{1/2}[(\mu^{-2} + \mu^{-1/2})n^{1+\varepsilon} + n^\varepsilon] + (\mu^{-3/2} + \log \mu)n^{1+\varepsilon} + \mu^{1/2}n^\varepsilon \right] \\ &\ll_\varepsilon \mu n^\varepsilon + \mu^{1/2} \log \mu n^{1+\varepsilon} + \mu^{-1} n^{1+\varepsilon} \leq (\mu + n\mu^{1/2} \log \mu) n^\varepsilon. \end{aligned}$$

Thus, we have shown

Theorem 5.2. *Let $\chi \in \widehat{\Gamma}$ be a Nebentypus character and $M = \Gamma_\chi \backslash \mathbb{H}$ a compact arithmetic hyperbolic surface ($\Gamma = \mathcal{R}(1)$). For $\mu \geq 1$ and any $n \in \mathbb{N}$ one has the bound*

$$(5.7) \quad K_{\tilde{s}_\mu \circ \mathcal{T}_n^\chi}(z, z) \ll_\varepsilon (\mu + n\mu^{1/2} \log \mu) n^\varepsilon, \quad \varepsilon > 0,$$

uniformly in $z \in M$.

□

Remark 5.3. The previous theorem is analogous to [16, Lemma 1.2]. Note that the bounds for the point pair invariants on \mathbb{H} used by Iwaniec and Sarnak in order to show [16, Lemma 1.2] are better than our bound (5.5) by a factor $(1 + u(\alpha \cdot z, z))^{-5/4}$ in the Stieltjes integral, but the lattice point counting function considered by them is unbounded, while ours is a priori bounded.

We can now deduce from Theorem 5.2 the following spherical subconvex bounds for Hecke–Maass forms with characters on arithmetic congruence surfaces.

Theorem 5.4. *For any $\phi_j \in L_\chi^2(\Gamma \backslash \mathbb{H})$ and $\varepsilon > 0$ one has*

$$\|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24} + \varepsilon}.$$

Proof. The assertion follows from Theorem 5.2 by arithmetic amplification. In fact, Theorem 5.4 corresponds to the spherical case of Theorem 5.7, of which a detailed proof will be given. For trivial χ , the assertion of the theorem is due to Iwaniec–Sarnak [16, Theorem 0.1]. Their method can also be used for non-trivial χ , as was done in [2, Section 10] for non-compact arithmetic surfaces. □

5.3. The non-spherical case. As above, let $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$, Γ be congruence arithmetic, and $\chi \in \widehat{\Gamma}$ a Nebentypus character. We shall now derive equivariant subconvex bounds for Hecke–Maass forms in $L^2_\chi(\Gamma \backslash G)$ and general K -types. From Corollary 4.3 one obtains for any $x \in \Gamma_\chi \backslash G$ and $\sigma \in \widehat{K}$ the estimate

$$K_{\mathcal{T}_n^\chi \circ \widetilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \mu M(x, n, \delta) + \mu^{1/2} \int_\delta^C s^{-1/2} dM(s) + \mu^{-1/2} \int_\delta^C s^{-1} dM(s),$$

where we set

$$M(\delta) := M(x, n, \delta) := \# \{ \alpha \in \Gamma_\chi \backslash \mathcal{R}(n) : u(xK, \alpha xK) < \delta \},$$

regarding $xK \in \Gamma_\chi \backslash G/K \simeq \Gamma_\chi \backslash \mathbb{H}$ as an element in \mathbb{C} , and took into account (5.4). In order to derive a uniform bound for $K_{\mathcal{T}_n^\chi \circ \widetilde{s}_\mu \circ \Pi_\sigma}(x, x)$, note that by Lemma 5.1 one has with $N(s) := s^{-1/2}$ and $\delta = \mu^{-1}$

$$\begin{aligned} \int_\delta^C s^{-1/2} dM(s) &= \underbrace{N(C)M(C)}_{\ll n^{1+\varepsilon}} - \underbrace{N(\delta)M(\delta)}_{\ll \delta^{-1/2}[(\delta + \delta^{1/4})n^{1+\varepsilon} + n^\varepsilon]} - \underbrace{\int_\delta^C M(s)N'(s) ds}_{\ll (s^{1/2} + s^{-1/4})n^{1+\varepsilon} - s^{-1/2}n^\varepsilon|_\delta^C} \\ &\ll_\varepsilon n^{1+\varepsilon} + \mu^{1/2}n^\varepsilon + \mu^{1/4}n^{1+\varepsilon} + \mu^{1/2}n^\varepsilon \ll (\mu^{1/2} + n\mu^{1/4})n^\varepsilon, \end{aligned}$$

while with $\widetilde{N}(s) := s^{-1}$ one computes

$$\begin{aligned} \int_\delta^C s^{-1} dM(s) &= \underbrace{\widetilde{N}(C)M(C)}_{\ll n^{1+\varepsilon}} - \underbrace{\widetilde{N}(\delta)M(\delta)}_{\ll \delta^{-1}[(\delta + \delta^{1/4})n^{1+\varepsilon} + n^\varepsilon]} - \underbrace{\int_\delta^C M(s)\widetilde{N}'(s) ds}_{\ll (\log s + s^{-3/4})n^{1+\varepsilon} - s^{-1}n^\varepsilon|_\delta^C} \\ &\ll_\varepsilon n^{1+\varepsilon} + \mu n^\varepsilon + \mu^{3/4}n^{1+\varepsilon} + (\log \mu)n^{1+\varepsilon} \ll (\mu + n\mu^{3/4})n^\varepsilon. \end{aligned}$$

Taking everything together we have shown

Theorem 5.5. *For any Nebentypus character $\chi \in \widehat{\Gamma}$, K -type $\sigma \in \widehat{K}$, $n \in \mathbb{N}$, and $\mu > 0$ we have*

$$K_{\mathcal{T}_n^\chi \circ \widetilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll (\mu + n\mu^{3/4})n^\varepsilon$$

uniformly in $x \in M = \Gamma_\chi \backslash G$. □

To deduce from Theorem 5.5 equivariant subconvex bounds, we shall make use of arithmetic amplification, following the original approach of Iwaniec and Sarnak [16]. Thus, let $\chi \in \widehat{\Gamma}$, $\sigma \in \widehat{K}$ be fixed, and $\{\phi_j\}_{j \in \mathbb{N}}$ be the orthonormal basis in $L^2(\Gamma_\chi \backslash G)$ introduced in (5.2) consisting of Hecke–Maass forms with Hecke eigenvalues $\lambda_j(n)$. Writing

$$\eta_j(n) := \frac{\lambda_j(n)}{\sqrt{n}}$$

we deduce with (3.7), (4.3), (5.1), and (5.2) that⁶

$$\begin{aligned} &\sum_{j \geq 0, \phi_j \in L^2_\sigma(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) \phi_j(z) \overline{\phi_j(w)} \eta_j(m) \eta_j(n) \\ &= \sum_{d|(n, m)} \sum_{j \geq 0, \phi_j \in L^2_\sigma(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) \phi_j(z) \overline{\phi_j(w)} \chi(d) \eta_j\left(\frac{nm}{d^2}\right) = \sum_{d|(n, m)} \frac{d\chi(d)}{\sqrt{nm}} K_{\mathcal{T}_{nm/d^2}^\chi \circ \widetilde{s}_\mu \circ \Pi_\sigma}(z, w). \end{aligned}$$

⁶Since the T_n^χ are adjoint operators for all n with $(n, q) = 1$, one has $\eta_j(n) = \overline{\eta_j(n)}$, compare [23, Theorem 5.3.8]. More generally, $(T_{\Gamma_\alpha \Gamma}^\chi)^* = T_{\Gamma_{\alpha' \Gamma}}^\chi$, where $\alpha' = \det(\alpha)\alpha^{-1}$.

Applying Theorem 5.5 we therefore obtain for arbitrary $N \in \mathbb{N}$

$$\begin{aligned} \sum_{j \geq 0, \phi_j \in L_\sigma^2(\Gamma_X \backslash G)} \varrho(\mu - \mu_j) |\phi_j(z)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 &= \sum_{n, m \leq N} \sum_{d|(n, m)} \frac{\chi(d)}{d} \sqrt{nm} z_n \overline{z_m} K_{\tilde{s}_\mu \circ \Pi_\sigma \circ \mathcal{T}_{nm/d^2}^\chi}(z, z) \\ &\ll_\varepsilon N^\varepsilon \sum_{n, m \leq N} \sum_{d|(n, m)} \frac{d}{\sqrt{nm}} |z_n z_m| \left(\mu + \frac{nm}{d^2} \mu^{3/4} \right), \end{aligned}$$

where $z_n \in \mathbb{C}$ are arbitrary complex numbers. A simple computation then gives

$$(5.8) \quad \sum_{j \geq 0, \phi_j \in L_\sigma^2(\Gamma_X \backslash G)} \varrho(\mu - \mu_j) |\phi_j(z)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \ll_\varepsilon N^\varepsilon \left[\mu \sum_{n \leq N} |z_n|^2 + N \mu^{3/4} \left(\sum_{n \leq N} |z_n| \right)^2 \right].$$

We thus arrive at

Theorem 5.6. *For any $\mu > 0$ and $N \in \mathbb{N}$ one has the estimate*

$$\sum_{\substack{\mu \leq \sqrt{\lambda_j} \leq \mu+1, \\ \phi_j \in L_\sigma^2(\Gamma_X \backslash G)}} |\phi_j(z)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \ll_\varepsilon N^\varepsilon \left[\mu \sum_{n \leq N} |z_n|^2 + N \mu^{3/4} \left(\sum_{n \leq N} |z_n| \right)^2 \right].$$

Proof. As explained at the end of Section 3, the test function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$ can be chosen such that $\varrho > 0$ on $[-1, 1]$. The theorem now follows from (5.8) and the estimate

$$\begin{aligned} \sum_{\substack{\mu \leq \sqrt{\lambda_j} \leq \mu+1, \\ \phi_j \in L_\sigma^2(\Gamma_X \backslash G)}} |\phi_j(z)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \cdot \min \{ \varrho(\mu) \mid \mu \in [-1, 1] \} \\ \leq \sum_{j \geq 0, \phi_j \in L_\sigma^2(\Gamma_X \backslash G)} \varrho(\mu - \mu_j) |\phi_j(z)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2. \end{aligned}$$

□

Next, one proceeds as follows. Let $j_0 \geq 0$ be fixed such that $\phi_{j_0} \in L_{\sigma, \chi}^2(\Gamma \backslash G)$, and consider the *amplifier*

$$z_n := \begin{cases} \eta_{j_0}(p), & n = p \leq \sqrt{N}, \\ -1 & n = p^2 \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

where p is a prime not dividing q . Note that (2.10) and (5.1) imply

$$\eta_j(p)^2 - \eta_j(p^2) = \begin{cases} 1 & \text{if } \phi_j \in L_\chi^2(\Gamma \backslash G), \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\left| \sum_{n \leq N} z_n \eta_{j_0}(n) \right| = \sum_{p \leq \sqrt{N}, p \nmid q} 1 = O(\sqrt{N} / \log N^{1/2}),$$

by the Prime Number Theorem. Writing $\lambda_j = 1/4 + r_j^2$ and taking $\mu = r_{j_0}$ Theorem 5.6 then gives

$$|\phi_{j_0}(z)|^2 \ll_\varepsilon N^{\varepsilon-1} \left[r_{j_0} \left(\sum_{p \leq \sqrt{N}} |\eta_{j_0}(p)|^2 + \sqrt{N} \right) + N r_{j_0}^{3/4} \left(\sum_{p \leq \sqrt{N}} |\eta_{j_0}(p)| + N^{1/2} \right)^2 \right].$$

As a next step, note that Jacquet-Langlands correspondence [17] and the study of Rankin-Selberg convolutions ([15, Theorem 8.3] and [8, Proposition 19.6]) imply for any $j \in \mathbb{N}$ with $\phi_j \in L_{\sigma, \chi}^2(\Gamma \backslash G)$ the bound

$$(5.9) \quad \sum_{n \leq N} |\eta_j(n)|^2 \ll_\varepsilon N^{1+\varepsilon} r_j^\varepsilon,$$

where n moves over natural numbers prime to q . Here we used the facts that the Strong Multiplicity One Theorem holds for $\mathrm{GL}(2)$ and each automorphic representation factors as a tensor product of local representations. Consequently, with (5.9) and Cauchy's inequality one deduces

$$|\phi_{j_0}(z)|^2 \ll_{\varepsilon} N^{\varepsilon} r_{j_0}^{\varepsilon} (r_{j_0} N^{-1/2} + r_{j_0}^{3/4} N).$$

Choosing $N = r_{j_0}^{1/3}$ finally gives

$$|\phi_{j_0}(z)|^2 \ll_{\varepsilon} r_{j_0}^{\frac{11}{12} + \varepsilon}$$

uniformly in $z \in \Gamma \backslash \mathbb{H}$. Thus, we have shown our first main result. For trivial σ and χ , this theorem is due to Iwaniec-Sarnak [16, Theorem 0.1].

Theorem 5.7. *For any ϕ_j in $L_{\sigma, \chi}^2(\Gamma \backslash G)$ and any $\varepsilon > 0$ one has*

$$\|\phi_j\|_{\infty} \ll_{\varepsilon} \lambda_j^{\frac{11}{48} + \varepsilon}.$$

□

6. EQUIVARIANT SUBCONVEX BOUNDS FOR SEMISIMPLE ALGEBRAIC GROUPS

In this section, we shall derive equivariant subconvex bounds for a large class of semisimple algebraic groups, generalizing the work of Marshall [21] to non-spherical situations. For this, let us return to the general setting of Sections 2 and 3. Let $L_{\sigma, \chi}^2(\Gamma \backslash G)$ be defined as in (4.1), and consider on this space the family of Hecke operators $T_{\Gamma \beta \Gamma}^{\chi}$ introduced in (2.7), together with the corresponding \mathbb{C} -module

$$H_{\Xi}^{\chi} := \langle T_{\Gamma \beta \Gamma}^{\chi} \mid \beta \in \Xi \rangle$$

generated by them. In what follows, we assume that there exists a submodule \mathcal{H} of H_{Ξ}^{χ} such that there exists an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2(\Gamma_{\chi} \backslash G)$ compatible with the decompositions (2.6) and (3.5) consisting of simultaneous eigenfunctions for P and all $\mathcal{T} \in \mathcal{H}$. We also suppose that \mathcal{T}^* belongs to \mathcal{H} for each $\mathcal{T} \in \mathcal{H}$ and that the cospheres $S_x^* M := \{(x, \xi) \in T^* M \mid p(x, \xi) = 1\}$ are strictly convex. As before, such simultaneous eigenfunctions will be called *Hecke–Maass forms*. Further, for $x \in \Gamma_{\chi} \backslash G$ and $\beta \in \Xi$ consider the lattice point counting function $M(x, \beta, \delta)$ defined in (4.4). We then have the following

Lemma 6.1. *Let ϕ_{j_0} be a Hecke–Maass form in $L_{\sigma, \chi}^2(\Gamma \backslash G)$ with corresponding spectral eigenvalue λ_{j_0} and q a suitable natural number. Assume that for each prime number p with $(p, q) = 1$ there exists a Hecke operator $\mathcal{T}'_p \in \mathcal{H}$ satisfying $\mathcal{T}'_p \phi_{j_0} = \phi_{j_0}$, and write $\mathcal{T}'_N := \sum_{p \leq N, (p, q) = 1} \mathcal{T}'_p$ for any $N \in \mathbb{N}$. As a linear operator in $L_{\sigma, \chi}^2(\Gamma \backslash G)$, $\mathcal{T}'_N \circ (\mathcal{T}'_N)^*$ can be represented as*

$$\mathcal{T}'_N \circ (\mathcal{T}'_N)^* = \sum_{u=1}^l a_u \mathcal{T}_{\Gamma \alpha_u \Gamma}^{\chi}$$

for certain $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in \Xi$.⁷ Further, suppose that there exist numbers $0 < \kappa \ll 1$ and $A_1, A_2 > 2$ such that for each $N \gg 1$ one has

$$(6.1) \quad \sum_{u=1}^l |a_u| |M(x, \alpha_u, N^{-A_2})| \ll N^{2-2\kappa}, \quad \sum_{u=1}^l |a_u| |\Gamma_{\chi} \backslash \Gamma \alpha_u \Gamma| \ll N^{A_1}.$$

Then, there exists a constant $\delta > 0$ such that

$$\|\phi_{j_0}\|_{\infty} \ll \lambda_{j_0}^{\frac{d - \dim K - 1}{2m} - \delta}.$$

⁷The existence of the l , a_u , and α_u is ensured by (2.3) and the assumptions about \mathcal{H} , though they might not be unique. Please delete the number (2.3) when you delete this footnote.

Proof. Set $\mu := \sqrt[\mathfrak{w}]{\lambda_{j_0}}$,

$$P_{N,q} := \#\{p \mid p \text{ is a prime, } 1 \leq p \leq N, (p, q) = 1\},$$

and denote by $\lambda'_{j,N}$ the eigenvalue of \mathcal{T}'_N for ϕ_j , so that

$$|\mathcal{T}'_N \phi_j(x)|^2 = \lambda'_{j,N} \overline{\lambda'_{j,N}} \phi_j(x) \overline{\phi_j(x)} = \mathcal{T}'_N \circ (\mathcal{T}'_N)^* \phi_j(x) \overline{\phi_j(x)}.$$

Using Corollary 4.3 one gets with $\mu_j = \sqrt[\mathfrak{w}]{\lambda_j}$ for any $x \in \Gamma_\chi \backslash G$

$$\begin{aligned} (P_{N,q})^2 |\phi_{j_0}(x)|^2 &= |\mathcal{T}'_N \phi_{j_0}(x)|^2 \leq \sum_{\substack{\mu \leq \mu_j \leq \mu+1, \\ \phi_j \in L^2_{\sigma}(\Gamma_\chi \backslash G)}} |\mathcal{T}'_N \phi_j(x)|^2 \ll \sum_{\substack{j \geq 0, \\ \phi_j \in L^2_{\sigma}(\Gamma_\chi \backslash G)}} \varrho(\mu - \mu_j) |\mathcal{T}'_N \phi_j(x)|^2 \\ &= K_{\mathcal{T}'_N \circ (\mathcal{T}'_N)^* \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \mu^{d-\dim K-1} N^{2-2\kappa} + \mu^{\frac{d-\dim K-1}{2}} N^A + \mu^{\frac{d-\dim K-1}{2}-1} N^{A'}, \end{aligned}$$

where we used a similar argument than in the proof of Theorem 5.6, $A = \frac{d-1}{2}A_2 + A_1$, and $A' = \frac{d+1}{2}A_2 + A_1$. Hence, the assertion follows from the Prime Number Theorem by taking $N \sim \mu^{\min(B, B')}$ where $B = \frac{d-\dim K-1}{2(A-2+2\kappa)}$ and $B' = \frac{d-\dim K+1}{2(A'-2+2\kappa)}$. \square

Remark 6.2. The assumptions of the previous lemma are primarily motivated by the work of Marshall [21] in the case that χ is trivial. One can easily verify that they are fulfilled in the setup of Section 5. Indeed, for each prime p with $(p, q) = 1$, there exists an element β_p in $\mathcal{R}(p^2)$ such that $\tilde{\mathcal{T}}_{p^2}^\chi = \mathcal{T}_{\Gamma\beta_p\Gamma}^\chi = \mathcal{T}_{p^2}^\chi - \mathcal{T}_{\Gamma(pI_2)\Gamma}^\chi$, see [23, p. 217]. Now, let $\tilde{\lambda}_j(p^2)$ be the eigenvalue of $\tilde{\mathcal{T}}_{p^2}^\chi$ belonging to the eigenfunction $\phi_j \in L^2_{\sigma, \chi}(\Gamma \backslash G)$. For each ϕ_j in $L^2_{\sigma, \chi}(\Gamma \backslash G)$ and p as above, $|\lambda_j(p)| \leq p^{1/2}/2$ implies that $|\tilde{\lambda}_j(p^2)| \geq p/2$ by $|\tilde{\lambda}_j(p^2)| = |\lambda_j(p)^2 - (p+1)\chi(p)|$. Therefore, if we choose

$$\mathcal{T}'_p := \begin{cases} \frac{1}{\lambda_{j_0}(p)} \mathcal{T}_p^\chi & \text{if } |p^{-1/2}\lambda_{j_0}(p)| > 1/2, \\ \frac{1}{\lambda_{j_0}(p^2)} \tilde{\mathcal{T}}_{p^2}^\chi & \text{otherwise,} \end{cases}$$

the mentioned assumptions must hold by Lemma 5.1. This choice is essentially the same as in the work of Blomer-Maga [3, 4] on $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{PGL}(n, \mathbb{R})$ in the case $n = 2$.

In what follows, we shall deduce equivariant subconvex bounds for the trivial character $\chi = 1$ by using Lemma 6.1. Thus, let H be a connected semisimple algebraic group of adjoint type over \mathbb{Q} , and write $H(F)$ for the set of F -rational points in H for a field F of characteristic zero. Let K be a maximal connected compact subgroup in $H(\mathbb{R})$. We omit an explanation for the condition “ K -small” in [21] here, but to be fulfilled, it is sufficient that $H(\mathbb{R})$ is isogeneous to a product of split classical groups or the split form of G_2 [21, Theorem 1.1]. The condition “complex” of [21] implies that there exist a CM extension F/F^+ and a connected semisimple group H^0/F^+ such that $H^0(F_v^+)$ is compact for each real place v of F^+ and $H = \mathrm{Res}_{F/\mathbb{Q}} H^0$, where F_v^+ is the completion of F^+ by v , and $\mathrm{Res}_{F/\mathbb{Q}}$ means the restriction of scalars from F to \mathbb{Q} . Finally, let K_0 be an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$. As the second main result of this paper we obtain

Theorem 6.3. *Let $G := H(\mathbb{R})$, $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$, and assume that each of the connected components of G contains points of Γ . Further, suppose that $H(\mathbb{Q}) \backslash H(\mathbb{A})$ is compact, so that $\Gamma \backslash G$ is also compact. Then, the following statements hold:*

- (1) *There is a submodule \mathcal{H} of H_Ξ^χ with $\chi = 1$, $\Xi = H(\mathbb{Q})$ such that there exists an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2_\sigma(\Gamma \backslash G)$ which consists of simultaneous eigenfunctions for P and all $\mathcal{T} \in \mathcal{H}$.*
- (2) *If H satisfies the condition “ K -small” or “complex” of [21] and the cospheres*

$$S_x^* M := \{(x, \xi) \in T^* M \mid p(x, \xi) = 1\}$$

are strictly convex, there exists a constant $\delta > 0$ such that for each ϕ_j with spectral eigenvalue λ_j one has

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-\dim K-1}{2m}-\delta},$$

where m denotes the order of P .

Proof. It is known that G has a finite number of connected components [24, Corollary 1], and we write G^0 for the identity component in G , as well as $\Gamma^0 := \Gamma \cap G^0$. The component G^0 (resp. Γ^0) is a normal subgroup of G (resp. Γ), and by assumption we have the isomorphism $\Gamma^0 \backslash \Gamma \cong G^0 \backslash G$ given by $\Gamma^0 \gamma \mapsto G^0 \gamma$. Hence, $L^2(\Gamma \backslash G)$ is isomorphic to $L^2(\Gamma^0 \backslash G^0)$ via the restriction of functions in $L^2(\Gamma \backslash G)$ to G^0 . In addition, one can show that for each double coset $\Gamma \alpha \Gamma$ with $\alpha \in H(\mathbb{Q}) \subset C(\Gamma)$ there exist elements $\alpha_1, \dots, \alpha_m$ in $H(\mathbb{Q}) \cap G^0 \subset C(\Gamma^0)$ such that

$$T_{\Gamma \alpha \Gamma} f = [\Gamma : \Gamma^0]^{-1} \sum_{j=1}^m T_{\Gamma^0 \alpha_j \Gamma^0} f, \quad \Gamma \alpha \Gamma = \bigsqcup_{j=1}^m \gamma_j \Gamma^0 \alpha_j \Gamma^0$$

for some γ_j in Γ and $m \leq [\Gamma : \Gamma^0]^2$. Notice that it is enough to consider $T_{\Gamma \alpha \Gamma}$ for $\alpha \in H(\mathbb{Q})$ in order to use the results in [21], and that Hecke operators are defined as in Section 2 even if G is not connected. Now, by translating the results in [21] to our non-adelic setting, in particular [21, Propositions 2.2 and 2.5] for the “ K -small” case and [21, Propositions 3.1 and 3.2] for the “complex” case, one verifies that the assumptions of Lemma 6.1 are fulfilled under the hypothesis of the theorem. Note that it is unnecessary to relate the subgroup K to the specific maximal connected compact subgroup $\underline{K}(\mathbb{R})$ considered in [21], because the assumptions in question are concerned only with the structure of the Hecke algebra and the lattice point counting function $M(x, \alpha, \delta)$.

Let us explain this in a more detailed way. By the finiteness of class numbers of algebraic groups [24, Theorem 5.1, Theorem 8.1] there exist elements $x_1 = 1, x_2, \dots, x_r$ in $H(\mathbb{A}_{\text{fin}})$ such that

$$(6.2) \quad H(\mathbb{A}) = \bigsqcup_{l=1}^r H(\mathbb{Q}) x_l H(\mathbb{R}) K_0.$$

Consequently,

$$H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0 \cong \bigsqcup_{l=1}^r \Gamma_l \backslash G, \quad \Gamma_l := H(\mathbb{Q}) \cap (H(\mathbb{R}) x_l^{-1} K_0 x_l).$$

Now, any function φ in $L^2_\sigma(\Gamma \backslash G)$ can be identified with a function $\varphi_{\mathbb{A}}$ in $L^2_\sigma(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0)$ by setting

$$\varphi_{\mathbb{A}}(\gamma x_l g k) := \begin{cases} \varphi(g) & \text{if } l = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \gamma \in H(\mathbb{Q}), \quad g \in G, \quad k \in K_0.$$

For each double coset $K_0 \alpha K_0$ with $\alpha \in H(\mathbb{A}_{\text{fin}})$, a linear operator $T_{K_0 \alpha K_0}$ on $L^2_\sigma(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0)$ is then defined by setting

$$(T_{K_0 \alpha K_0} \phi_{\mathbb{A}})(x) := \sum_{h \in K_0 \alpha K_0 / K_0} \phi_{\mathbb{A}}(xh), \quad \phi_{\mathbb{A}} \in L^2(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0).$$

Moreover, there exist finitely many elements β_1, \dots, β_m in $H(\mathbb{Q})$ such that

$$(6.3) \quad H(\mathbb{Q}) \cap (H(\mathbb{R}) K_0 \alpha^{-1} K_0) = \bigsqcup_{i=1}^m \Gamma \beta_i \Gamma.$$

In view of (6.2) and (6.3) it follows that for all $\varphi \in L^2_\sigma(\Gamma \backslash G)$

$$(6.4) \quad T_{K_0 \alpha K_0} \varphi_{\mathbb{A}} = \sum_{i=1}^m T_{\Gamma \beta_i \Gamma} \varphi.$$

Hence, any adelic Hecke operator $T_{K_0 \alpha K_0}$ can be regarded as a sum of non-adelic Hecke operators via the identification $\varphi \equiv \varphi_{\mathbb{A}}$. In order to apply Lemma 6.1 in the present context, choose $\chi = 1$, $\Xi = H(\mathbb{Q})$, and put

$$\mathcal{H} := \langle T_{K_0 \alpha K_0} \mid \alpha \in H(\mathbb{Q}_{p'}), \quad p' \text{ prime}, \quad (p', q) = 1 \rangle$$

for a suitable large natural number q . Note that by [23, Proof of Theorem 2.8.2 (2)] we have $(T_{K_0 \alpha K_0})^* = T_{K_0 \alpha^{-1} K_0} \in \mathcal{H}$. Since each automorphic representation of $H(\mathbb{A})$ factors [9] as a tensor product of irreducible unitary representations of $H(\mathbb{R})$ and $H(\mathbb{Q}_{p'})$ for all primes p' , there exists an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2_\sigma(\Gamma \backslash G)$ consisting of simultaneous eigenfunctions for P and all $\mathcal{T} \in \mathcal{H}$.

Now, let j_0 be fixed. Applying the results in [21] to the function $\psi := \phi_{j_0, \mathbb{A}}$, that is also denoted by ψ there, one can verify the assumptions of Lemma 6.1 for ϕ_{j_0} . Indeed, by [21, Propositions 2.2 and 3.1], for each prime p there exists a Hecke operator $\mathcal{T}_p \in \mathcal{H}$ such that $\mathcal{T}_p \psi = \psi$ holds and \mathcal{T}_p is a linear combination of operators $T_{K_0 \alpha K_0}$ with $\alpha \in H(\mathbb{Q}_p)$. In view of (6.4) we can identify \mathcal{T}_p with a non-adelic Hecke operator \mathcal{T}'_p on $L^2_\sigma(\Gamma \backslash G)$ such that

$$\mathcal{T}'_p \phi_{j_0} = \mathcal{T}_p \phi_{j_0, \mathbb{A}} = \phi_{j_0, \mathbb{A}} = \phi_{j_0}.$$

Similarly, set $\mathcal{T}_N := \sum_{p \leq N, (p, q)=1} \mathcal{T}_p$, and denote the corresponding non-adelic Hecke operators by \mathcal{T}'_N . By the convolution on $H(\mathbb{A}_{\text{fin}})$, there exist $n \in \mathbb{N}$, $b_k \in \mathbb{C}$, and $\omega_k \in H(\mathbb{A}_{\text{fin}})$ such that

$$\mathcal{T}_N \circ (\mathcal{T}_N)^* = \sum_{k=1}^n b_k T_{K_0 \omega_k K_0}.$$

The corresponding $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in C(\Gamma)$ in the decomposition of $\mathcal{T}'_N \circ (\mathcal{T}'_N)^*$ in Lemma 6.1 are then obtained from this equality via the identification (6.4). Finally, the upper bounds (6.1) in Lemma 6.1 can be verified using (6.3), (6.4) and the arguments in [21, Sections 2 and 3], completing the proof of the theorem. \square

Example 6.4 (Subconvex bounds for $\text{PGL}(n, \mathbb{R})$). Choose a central division algebra D of index n over \mathbb{Q} such that $D \otimes \mathbb{R} \cong M(n, \mathbb{R})$. It is well-known that the equivalence classes of central division algebras over \mathbb{Q} are parameterized by the Brauer group $\text{Br}(\mathbb{Q})$, which can be realized as the set

$$\left\{ (a, x) \mid a \in \{0, 1/2\}, \quad x = (x_p)_p \in \bigoplus_p \mathbb{Q}/\mathbb{Z}, \quad a + \sum_p x_p = 0 \pmod{\mathbb{Z}} \right\}$$

via the Brauer-Hasse-Noether theorem, compare [24, Theorem 1.12 on p. 38]. If we choose a prime p_1 and a parameter $(0, x)$ in $\text{Br}(\mathbb{Q})$ such that $x_{p_1} = a/n$ and a is prime to n , then there is a central division algebra D corresponding to $(0, x)$ and satisfying $D \otimes \mathbb{R} \cong M(n, \mathbb{R})$. A semisimple algebraic group H of adjoint type over \mathbb{Q} is then defined by $H := \text{Aut}_{\text{alg}}(D) = \text{PGL}(1, D)$, compare [18, p. 328 and (23.1) on p. 344]. Clearly, $G := H(\mathbb{R}) \cong \text{PGL}(n, \mathbb{R})$, and $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$ is cocompact for any open compact subgroup K_0 of $G(\mathbb{A}_{\text{fin}})$, while H satisfies the “ K -small” condition by [21, Theorem 1.1]. It is known that G is connected if n is odd, but we have $[G : G^0] = 2$ if n is even. Hence, in the case n is even, we suppose Γ contains an element which belongs to $G \setminus G^0$. Writing q for the integer Q given in [21, p. 8 and p. 11], the assumptions of Theorem 6.3 hold for G , q , and any Hecke–Maass form ϕ_j in $L^2_\sigma(\Gamma \backslash G)$ with eigenvalue λ_j , yielding the sub-convex bound

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{n^2+n-4}{4m} - \delta}$$

for some $\delta > 0$.

Example 6.5 (Subconvex bounds for $\text{PGL}(n, \mathbb{C})$). For an imaginary quadratic field F over \mathbb{Q} there is a central division algebra D of index n over F . Assume that there exists a unitary involution ι on D , that is, an involution of the second kind, and consider the projective unitary group over \mathbb{Q} given by $H^0 := \text{Aut}_F(D, \iota) = \text{PGU}(D, \iota)$, see [18]. If we assume that $H^0(\mathbb{R})$ is compact, then the semisimple algebraic group $H := \text{Res}_{F/\mathbb{Q}} \text{PGU}(D, \iota)/F$ satisfies the condition “complex”. Since $H(\mathbb{R})$ is isomorphic to $\text{PGL}(n, \mathbb{C})$, Theorem 6.3 yields subconvex bounds for $\Gamma \backslash \text{PGL}(n, \mathbb{C})$, where Γ is defined by an open compact subgroup K_0 in $H(\mathbb{A}_{\text{fin}})$ in the same manner.

Note that any central division algebra over a totally real field provides an example similar to Example 6.4, while in Example 6.5 we may replace an imaginary quadratic field over \mathbb{Q} by a CM extension. Further examples in the “complex” case can be obtained by considering a skew field with an orthogonal involution.

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